On the global Hadamard condition in QFT and the signed squared geodesic distance defined in domains larger than convex normal neighbourhoods

Valter Moretti

Department of Mathematics, University of Trento, and INFN-TIFPA
via Sommarive 14, I-38123, Povo (Trento), Italy.
valter.moretti@unitn.it

July, 2021

Abstract

We consider the global Hadamard condition in algebraic QFT in curved spacetime, pointing out the existence of a technical problem in the literature concerning well-posedness of the global Hadamard parametrix in normal neighbourhoods of Cauchy surfaces. We discuss in particular the definition of the (signed) geodesic distance σ and related structures in an open neighbourhood of the diagonal of $M \times M$ larger than $U \times U$, for a normal convex neighborhood U, where (M, g) is a Riemannian or Lorentzian (smooth Hausdorff paracompact) manifold. We eventually propose a quite natural solution which slightly changes the original definition by B.S. Kay and R.M. Wald and relies upon some non-trivial consequences of paracompactness property. The proposed re-formulation is in agreement with M.J. Radzikowski's microlocal version of the Hadamard condition.

1 Introduction

The use of *Hadamard states* is nowadays pervasive in perturbative algebraic QFT (aQFT) in curved spacetime [BDFY15]. The rigorous definition of Hadamard state in terms of *short distance behaviour* of the two-point function was stated in the celebrated paper [KW91] by B.S. Kay and R.M. Wald for the first time. Some years later, that technically complex definition was translated into the language of microlocal analysis within a pair of nice papers by M.J. Radzikowski [Rad96a, Rad96b]. The original geometric definition of [KW91] has been exploited for instance to deal with rigorous intepretations of the Hawking radiation. See [MP12], and the recent interesting paper [KPV21].

Locally (and a bit roughly) speaking, in a globally-hyperbolic four-dimensional spacetime (M, g), an algebraic state ω of a real Klein-Gordong quantum field is of *Hadamard type* if its *two-point function* Λ_ω has the *Hadamard short-distance singularity*,

$$\Lambda_\omega(x,y) = \frac{1}{(2\pi)^2}\left(\frac{\Delta(x,y)^{1/2}}{\sigma(x,y)} + v(x,y)\ln\sigma(x,y)\right) + H_\omega(x,y) \tag{1}$$

when viewed as an integral kernel (see Section 3 for some technical details here disregarded). H_ω is a smooth function depending on the state ω, whereas Δ, v, σ are universal geometric objects constructed out of the local geometry only. In particular, $\sigma(x, y)$ is the so-called *signed squared geodesic distance* of $x, y \in M$. It is defined as the squared length – with the appropriate sign – of *the* geodesic segment joining x and y. The so-called *Hadamard parametrix* is the singular universal part $\Lambda_\omega(x, y) - H(x, y)$.

Since there are many geodesics, in principle, joining x and y, a standard possibility is to assume that the identity above is true in a *normal convex neighbourhood*[1] (see Section 2). It is an open set U such that every pair of points $x, y \in U$ can be joined by a *unique* geodesic segment $\gamma : [0,1] \to M$ *that belongs to the set*: $\gamma([0,1]) \subset U$. This elementary precaution is not enough however in the global definition discussed in [KW91]. It is because (1) is assumed to be valid for pairs (x,y) contained in *many* normal convex neighbourhoods. In principle this gives rise to a cumbersome many-valued function σ. This is one of the most difficult technical issues tackled in [KW91]. If x and y are *causally related*, a natural choice of U for the given x, y exists which solves the problem of the definition of $\sigma(x,y)$ and it was adopted in [KW91] (see Remark 10 below for more details). U, if exists, is a normal convex neighbourhood that *simultaneously includes* x, y *and their causal double cone* $J(x,y)$ (see (7)). Obviously, every other normal convex neighbourhood U' which both includes x, y and $J(x,y)$ must also contain the causal geodesic γ joining x and y in U. As U' is convex, γ is also *its own* geodesic γ' joining x and y in U'. There is, in fact, *only one* geodesic segment (parametrized in $[0,1]$) joining a pair of causally related points x and y in common for the subfamily of the said normal convex sets. For these pairs (x,y), $\sigma(x,y)$ can be therefore unambiguously defined.

Physics is properly reflected by the family of causal geodesic, but a mathematically coherent definition of the Hadamard parametrix needs to consider also non-causal geodesics: the non-causal ones "arbitrarily close" to the causal ones. For technical reasons, in [KW91] σ was therefore also required to be smooth and well-defined *in a neigbourhood* $\mathscr{O}$ of that special family of causally related pairs (x,y). We stress that the neighbourhood $\mathscr{O}$ may also contain non-causally connected pairs. The argument used to give a non-ambiguous definition of σ *cannot* be used for those pairs. The existence of $\mathscr{O}$ with a non-ambiguous extension of the definition of σ was assumed in [KW91] and also in [Rad96a, Rad96b] without a proof. In this author's view, it remains a gap in the whole construction. This work is devoted to that gap.

We shall not try to directly prove the existence of that $\mathscr{O}$. Our solution relies on a thin refinement of the definition of Hadamard parametrix which is possible thanks to a consequence of the paracompactness property. The final new definition of Hadamard state, which is a quite slight modification of the original definition in [KW91], though it is based on a non-trvial topological result, turns out to be in agreement with the microlocal version of the Hadamard condition.

To achieve our final goal, in the first part of the paper, we shall focus on the more abstract and mathematically-minded problem of a well-posed definition of σ (and related geometric objects) in a neighbourhood if the diagonal of $M \times M$. This issue is the core of the problem with the Hadamard parametrix, but it may have other applications in mathematical physics, so that it deserves a separate study.

A concrete elementary illustration of the problems one faces when trying to define $\sigma(x,x')$ in a non-trivial spacetime is the following one[2]. Consider the spacetime (M,g) constructed out of the $1+1$ Minkowski spacetime periodically identified under $(t,x) \mapsto (t, x+2L)$ $(c=1)$. Differently from the Minkowski space, M is not normal convex in its own right. *To define* $\sigma(x,x')$*, one is therefore forced to make a choice of a normal convex open set containing* x *and* x'*.*

[1] A weaker requirement is that the identity is valid in a normal neighborhood of one of the points.

[2] This illustration is a straightforward re-elaboration of an example proposed by C.J. Fewster to the author in a private communication.

Let $x = (0, L/2)$ and $x' = (L/2, L) \equiv (L/2, -L)$. These points are causally related and $J(x, x')$ (a null line segment) can be thickened up to become a normal convex neighbourhood U. We wish to define the function σ near (x, x'). Let us first consider nearby points that are still causally connected. In particular, let $y = x$ and $y' = (L/2, L - \epsilon) = (L/2, -L - \epsilon)$ with $0 < \epsilon \ll L$. These are causally related and near to (x, x'). In the considered case, we can also assume, enlarging U if necessary, that $U \supset J(y, y')$. We can create further normal convex neighbourhoods which include (y, y') by

(i) thickening the line segment (in $\mathbb{R}^2$) between $(0, L/2)$ and $(L/2, L - \epsilon)$ which is a timelike geodesic between y and y'. It produces a convex neighbourhood U' which we can assume to satisfy again $U' \supset J(y, y')$. The line segment between $(0, L/2)$ and $(L/2, L - \epsilon)$ belongs to both U and U'. Consequently, $\sigma(y, y')$ referred to U must coincide with $\sigma(y, y')$ referred to U'.

(ii) thickening the line segment between $(0, L/2)$ and $(L/2, -L - \epsilon)$ which is a spacelike geodesic between y and y'. In this case we obtain a value for $\sigma(y, y')$ different from the one computed in U.

This illustrates why, when defining $\sigma(y, y')$ in the causally connected case, the condition on the geodesically convex neighbourhood that it contains $J(y, y')$ permits to select a common notion of distance.

Next, consider the causally disconnected case. Let $y = x = (0, L/2)$ and $y' = (L/2, L + \epsilon) = (L/2, -L + \epsilon)$ with $0 < \epsilon \ll L$. These are not causally connected. We can still create normal convex neighbourhoods containing y and y' by thickening the line segment between $(0, L/2)$ and $(L/2, L + \epsilon)$ or the one between $(L/2, -L + \epsilon)$ and $(0, L/2)$ giving spacelike geodesics of differing lengths and different values of $\sigma(y, y')$. There are actually infinitely many other possibilities that can be obtained using other image points $(L/2, (2n + 1)L + \epsilon)$ $n \in \mathbb{Z}$ thoroughout a very slim thickening of these segments, wrapping on the cylinder without any self-intersection.

2 Extension of the signed squared geodesic distance and related structures

Smooth manifolds are hereafter assumed to be Hausdorff and paracompact[3]. We adopt the Lorentzian signature $(-, +, \cdots, +)$ and we follow [ON83] concerning basic defintions, notation and results in the theory of Lorentzian manifolds (see [Mi19] for an up-to-date general review).

2.1 Normal convex sets and the local definition of the (signed) squared geodesic distance σ

Following Chapter 5 of [ON83], $\exp : \mathcal{D} \subset TM \to M$ will denote the standard *exponential map* associated to the geodesic flow of g of a smooth Riemannian or Lorentzian manifold (M, g). Its maximal domain $\mathcal{D}$ is a fiberwise starshaped open neighborhood of the zero section of TM, and $\exp_p(v) := \exp(p, v)$ if $(p, v) \in \mathcal{D}$.

[3] For a topological space, to be homeomorphic to $\mathbb{R}^n$, Hausdorff, and 2nd countable imply paracompactness as is well known, whereas to be locally homeomorphic to $\mathbb{R}^n$, Hausdorff, and paracompact imply 2nd countablility if the space has countably many components [KN96].

Definition 1: If (M, g) is a Riemannian or Lorentzian smooth manifold, a **normal convex neighbourhood** U – also known as **normal convex open set** – is an open set $U \subset M$ such that, for every $q \in U$, there is a starshaped open neighbourhood $V_q^{(U)}$ of the origin of T_qM such that $\exp_q : V_q^{(U)} \to U$ is a diffeomorphism.

Remark 2: (1) U as above is **geodesically starshaped** with respect to every $p \in U$: for every other $q \in U$ there is only one geodesic segment $\gamma_{pq}^{(U)} : [0,1] \to M$ such that both $\gamma_{pq}^{(U)}(0) = p$, $\gamma_{pq}^{(U)}(1) = q$ and $\gamma_{pq}^{(U)}([0,1]) \subset U$ are valid (Proposition 31 in chapter 3 of [ON83]). By definition of $\exp_p$, it holds $\gamma_{pq}^{(U)}(t) = \exp_p(tv_q)$ where $v_q := \exp_p^{-1}(q)$ and $t \in [0,1]$.

(2) As U is geodesically starshaped with respect to every point $p \in U$, it is not difficult to prove that $V_p^{(U)}$ in Def. 1 is completely determined by U and $p \in U$.

(3) The set $\cup_{p\in U}\{p\} \times V_p^{(U)} \subset \mathcal{D}$ is open in TM. This is because the differential of the bijective map $U \times U \ni (p,q) \mapsto (p, (\exp_p|_{V_p^{(U)}})^{-1}(q)) \in \cup_{p\in U}\{p\} \times V_p$ is everywhere non-singular so that the map is open. $\exp : \cup_{p\in U}\{p\} \times V_p^{(U)} \to U \times U$ is the inverse diffeomorphism and thus it is smooth (see in particular Lemma 9, Chap 5 [ON83] and comments around it).

A crucial result by Whitehead[4] proves that (Propositition 7 chapter 5 of [ON83] and its proof.)

Theorem 3: *For a Riemannian or Lorentzian smooth manifold (M, g), the family of normal convex open sets is not empty and forms a topological basis of M.*

Among other important constructions, the Whitehead theorem and the properties of exp allow one to define the so-called **(signed) squared geodesic distance** also known as *Synge's world function.* If U is an open normal convex set in (M, g),

$$\sigma_U(p,q) := g_p\left(\dot\gamma_{pq}^{(U)}(0), \dot\gamma_{pq}^{(U)}(0)\right) = \pm\left(\int_0^1 \sqrt{\left|g\left(\dot\gamma_{pq}^{(U)}(t), \dot\gamma_{pq}^{(U)}(t)\right)\right|}dt\right)^2 \quad \text{for} \quad p,q \in U \tag{2}$$

where the sign $-$ appears only if g is Lorentzian and $\gamma_{pq}^{(U)}$ is timelike. This function is smooth in $U \times U$ because $\gamma_{pq}^{(U)}(t) = \exp_p\left(t(\exp_p|_{V_p^{(U)}})^{-1}(q)\right)$ is smooth in $[0,1] \times U \times U$, it being the restriction to $[0,1] \times U \times U$ of the composition of three smooth functions (defined on open sets): $\exp : \mathcal{D} \to M$, the multiplication with t, and a component of $U \times U \ni (p,q) \mapsto (p, (\exp_p|_{V_p^{(U)}})^{-1}(q)) \in \cup_{p\in U}\{p\} \times V_p$ according to (3) in Remark 2.

It is evident that $\sigma_U(p,q)$ strictly depends on the choice of the normal convex neighborhood containing the points p, q. If there were another normal convex neighborhood $U' \ni p, q$, in

[4] The definition of normal convex neighbourhoods and Whitehead's result are more generally true for smooth manifolds equipped with smooth affine connections [KN96], however in this paper we stick to the smooth Levi-Civita connection generated by g.

general $\sigma_U(p,q) \neq \sigma_{U'}(p,q)$ because the two sides refer to generally different geodesic segments: one stays in U and the other stays in U', though both geodesics join p and q. This fact prevents one from defining σ as a global smooth function over $M \times M$.

2.2 Assignment of geodesics around the diagonal of $M \times M$ and extension of σ thereon

A natural issue which pops out at this juncture is *whether or not σ can be more globally defined, at least in an open neighbourhood $\mathscr{A}$ of the* **diagonal** $\Delta_M := \{(p,p) \mid p \in M\}$ *of $M \times M$.*

The root of the problem is that, generally speaking, there are many geodesics connecting a pair of points p, q and $\sigma(p,q)$ depends on the choice of one of those curves. One restricts to work in a "small" neighbourhood $\mathscr{A}$ of the diagonal of $M \times M$ because it seems that the choice should be easier if p and q are close to each other. (There are however results concerning really global definitions of σ, on the whole $M \times M$, when assuming suitable hypotheses on the topology of M [CM04].) To address the issue above, one may therefore wonder *if it is possible to define a jointly smooth assignment of geodesic segments $\gamma_{pq}(t) = \Gamma(t,p,q)$ where $t \in [0,1]$, (p,q) varies in a neighborhood $\mathscr{A}$ of Δ_M and $\gamma_{pq}(0) = p$, $\gamma_{pq}(t) = q$.* Indeed, equipped with such an assignment, σ can be defined on $\mathscr{A}$ by direct use of (2).

Remark 4: If (M,g) is Riemannian and its *injectivity radius* is positive, then other known ways exist to define a smooth notion of (squared) geodesic distance in a neighbourhood of the diagonal of $M \times M$ (see, e.g.,[Sh91] for the case of a bounded geometry manifold in particular). However, we refer here to the general case where (M,g) may be Lorentzian, or Riemannian with zero injectivity radius.

An idea to construct Γ and σ in $\mathscr{A}$ (see also the discussion on p.131 of [ON83]) relies on the insight that sufficiently small normal convex neighbourhoods are expected to have intersections which are normal convex as well. In that case, if $U \cap U'$ is convex and both $p,q \in U$, $p,q \in U'$, then the unique geodesic segment $\Gamma_U(t,p,q) := \gamma^{(U)}_{p,q}(t) \in U$, $t \in [0,1]$, joining them in U coincides with the analogue $\Gamma_{U'}(t,p,q) := \gamma^{U'}_{p,q}(t) \in U'$ joining p and q in U', since it is the unique geodesic segment joining p and q in $U \cap U'$. Therefore $\Gamma_U(t,p,q) = \Gamma_{U \cap U'}(t,p,q) = \Gamma_{U'}(t,p,q)$. If a covering $\mathcal{C}$ of M exists made of normal convex open sets such that $U, U' \in \mathcal{C}$ implies that $U \cap U'$ is convex as well, then a jointly smooth assignment of geodesic segments $\Gamma : [0,1] \times \mathscr{A} \to M$ joining the arguments $(p,q) \in \mathscr{A}$ (i.e., $\Gamma(0,p,q) = p$ and $\Gamma(1,p,q) = q$) is well defined and smooth on the open neighbourhood $\mathscr{A}$ of Δ_M. It suffices to define

$$\mathscr{A} := \bigcup_{U \in \mathcal{C}} U \times U \tag{3}$$

$$\text{if } (x,y) \in \mathscr{A}, \quad \Gamma(t,x,y) := \Gamma_U(t,x,y) \quad \text{where } U \in \mathcal{C} \text{ is such that } x,y \in U. \tag{4}$$

Indeed, if $(x,y) \in \mathscr{A}$, then there must exist $U \in \mathcal{C}$ such that $x,y \in U$. Next, the right-hand side of (2) is well defined, since it does not depend on U if there are other elements in $\mathcal{C}$ containing x,y as pointed out above. Γ is also jointly smooth on $\mathscr{A}$ because it is locally jointly smooth. In this way, an associated signed squared geodesic distance $\sigma : \mathscr{A} \to \mathbb{R}$ results to be well-defined and smooth because composition of smooth functions:

$$\sigma(p,q) := g_p\left(\frac{\partial \Gamma}{\partial t}|_{(0,p,q)}, \frac{\partial \Gamma}{\partial t}|_{(0,p,q)}\right), \quad (p,q) \in \mathscr{A}\ . \tag{5}$$

Existence of the covering $\mathcal{C}$ is guaranteed when explicitly assuming Hausdorff and paracompactness hypotheses on M[5]. In fact, paracompactness possesses an important technical feature discovered by A.H. Stone [St49] (see also [Mi59]).

Theorem 5: *A topological space X is Hausdorff and paracompact if and only if it is T_1 and every covering $\mathcal{C}$ of X made of open sets admits a* ***-refinement** *of open sets. That is another covering $\mathcal{C}^*$ of open sets such that, for every $V \in \mathcal{C}^*$,*

$$\bigcup\{V' \in \mathcal{C}^* \mid V' \cap V \neq \varnothing\} \subset U_V \quad \text{for some } U_V \in \mathcal{C}.$$

(Notice that $V \subset U_V \in \mathcal{C}$ in particular, so that a *-refinement is a refinement as well). This theorem implies the existence of the desired well-behaved covering $\mathcal{C}$ of normal convex open sets of (M,g) (see also Lemma 10 in chapter 5 of [ON83]).

Proposition 6: *Let (M,g) be a smooth (Hausdorff paracompact) Riemannian or Lorentzian manifold and $\mathcal{A}$ a covering of M made of open sets (possibly $\mathcal{A} := \{M\}$). Then there exists a covering $\mathcal{C}$ of M sets such that,*

(a) $\mathcal{C}$ is a refinement of $\mathcal{A}$ (i.e., if $C \in \mathcal{C}$, then $C \subset U_C \in \mathcal{A}$) made of normal convex open sets;

(b) if $C, C' \in \mathcal{C}$ and $C \cap C' \neq \varnothing$, then $C \cap C'$ is a normal convex open set.

Proof. Using Theorem 8, consider the covering $\mathcal{C}_0$ made of all normal convex neighbourhoods that are subsets of the elements of $\mathcal{A}$. Exploiting Theorem 5, consider a refinement $\mathcal{C}_0^*$ of $\mathcal{C}_0$ satisfying, for every $V \in \mathcal{C}_0^*$,

$$\bigcup\{V' \in \mathcal{C}_0^* \mid V' \cap V \neq \varnothing\} \subset C_V \quad \text{for some } C_V \in \mathcal{C}_0.$$

The proof concludes by defining $\mathcal{C}$ as the family of normal convex neighbourhoods contained within elements of $\mathcal{C}_0^*$ so that (a) is in particular true by construction. To prove (b), we start by observing that, if $C, C' \in \mathcal{C}$, then $C \subset V$ and $C' \subset V'$ for some $V, V' \in \mathcal{C}_0^*$; if furthermore $C \cap C' \neq \varnothing$, we conclude that $V \cap V' \neq \varnothing$ and thus $C \cup C' \subset V \cup V' \subset C_V$. Property (b) now comes easily using convex normality of C_V. First the intersection $C \cap C'$ is open. Next, if $p, q \in C \cap C'$, then the unique geodesic segment $\gamma : [0,1] \to C_V$ joining p and q is also completely included in $C \cap C'$ since it must simultaneously stay in C and C', they being normal convex as well. As a consequence, if $p \in C \cap C'$, it necessarily holds $C \cap C' = \exp_p(V_p^{(C\cap C')})$ for some star-shaped open neighbourhood $V_p^{(C\cap C')} := (\exp_p|_{V_p^{(C)}})^{-1}(C \cap C')$ of the origin of T_pM. Notice that $\exp_p|_{V_p^{(C\cap C')}} : V_p^{(C\cap C')} \to C \cap C'$ is a diffeomorphism because it is the restriction of the diffeomorphism $\exp_p|_{V_p^{(C)}} : V_p^{(C)} \to C$. In summary, $C \cap C'$ fulfils Definition 1 and the proof is over. □

Definition 7: *If (M,g) is a smooth Riemannian or Lorentzian manifold, a* **strongly convex covering** *of M is a covering $\mathcal{C}$ of M made of normal convex open sets such that $C \cap C'$ is normal*

[5] This idea is sketched in Lemma 10 of chapter 5 of [ON83] unfortunately with very few details and without explicitly referring to the crucial topological result of Theorem 5.

convex if $C, C' \in \mathcal{C}$.

Collecting all results, we are in a position to state the main theorem of this section which also includes a (local) uniqueness statement.

Theorem 8: *Let (M, g) be a smooth (Hausdorff paracompact) Riemannian or Lorentzian manifold, and $\mathcal{C}$ a strongly convex covering of M. Then the following facts hold.*

(a) Defining the open neighbourhood $\mathscr{A} \supset \Delta_M$ as in (3) with respect to $\mathcal{C}$, the assigment of geodesic segments (2)

$$\Gamma : [0,1] \times \mathscr{A} \ni (t, p, q) \to \gamma_{p,q}(t) \in M \quad \text{where } \gamma_{p,q}(0) = p \text{ and } \gamma_{p,q}(1) = q,$$

and the (signed) squared geodesic distance (5)

$$\sigma(p, q) := g_p(\dot{\gamma}_{p,q}(0), \dot{\gamma}_{p,q}(0)) \quad \text{for } (p, q) \in \mathscr{A}$$

are well-defined and smooth on $\mathscr{A}$.

(b) If $\mathscr{A}' \supset \Delta_M$, $\Gamma' : [0,1] \times \mathscr{A}' \to M$, and $\sigma' : \mathscr{A}' \to \mathbb{R}$ is another triple as in (a) but constructed out of another strongly convex covering $\mathcal{C}'$, then there is an open set $\mathscr{A}'' \subset M \times M$ such that

$$\mathscr{A} \cap \mathscr{A}' \supset \mathscr{A}'' \supset \Delta_M \,, \quad \Gamma|_{[0,1]\times\mathscr{A}''} = \Gamma'|_{[0,1]\times\mathscr{A}''} \,, \quad \sigma|_{\mathscr{A}''} = \sigma'|_{\mathscr{A}''} \,. \tag{6}$$

Proof. (a) If $\mathscr{A}$ is as in (3), $\Gamma : [0,1] \times \mathscr{A} \to M$ defined as in (2) and $\sigma : \mathscr{A} \to \mathbb{R}$ defined as in (5) are well-defined and smooth as discussed in the paragraph before Eq. (3) and after Eq. (2). (b) Define a new covering $\mathcal{C}_1$ (a simultaneous refinement of $\mathcal{C}$ and $\mathcal{C}'$) made of the sets $C \cap C'$, for all choices of $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$. According to Proposition 6, define $\mathcal{C}''$ as a refinement of $\mathcal{C}_1$ made of normal convex neighborhoods such that $U, U' \in \mathcal{C}''$ implies that $U \cap U'$ is empty or normal convex and define $\mathscr{A}'' := \cup_{U'' \in \mathcal{C}''} U'' \times U''$. By construction, both $\mathscr{A}'' \subset \mathscr{A}$ and $\mathscr{A}'' \subset \mathscr{A}'$. Moreover, if $x, y \in \mathscr{A}''$ then we have both $x, y \in U'' \subset U \in \mathcal{C}$ and $x, y \in U'' \subset U' \in \mathcal{C}'$, so that $\Gamma(t, x, y) = \Gamma_U(t, x, y) = \Gamma_{U''}(t, x, y) = \Gamma_{U'}(t, x, y) = \Gamma'(t, x, y)$. The same fact holds true σ and σ' in view of their definition (5) in terms of Γ and Γ'. □

Definition 9: A triple $(\mathscr{A}, \Gamma, \sigma)$ as in in (a) of Theorem 8 is said to be **subordinated** to $\mathcal{C}$.

3 An issue with the global Hadamard condition

Before addressing a second issue still related to σ and associated structures, we summarize the relevant notions introduced by the milestone paper [KW91] where, for the first time, a rigorous definition of a Hadamard state was proposed and used by B.S. Kay and R.M. Wald. The definition was used in [KW91] (relying on previous work as [FSW78] and [Ka85]) to establish some important uniqueness results of QFT on a spacetime equipped with a bifurcate Killing horizon related to the KMS states of a real Klein-Gordon scalar field with the Hawking temperature. However, the definition of Hadamard state discussed therein applies to every (four-dimensional) globally hyperbolic spacetime.

3.1 Hadamard states according to [KW91]

If (M, g) is a time-oriented smooth spacetime and $x, y \in M$,

$$J(x,y) := (J^-(x) \cap J^+(y)) \cup (J^-(y) \cap J^+(x)) \,, \tag{7}$$

where $J^{\pm}(S)$ are defined as in [ON83]. We say that x, y are **causally related** in (M, g) if $J(x, y) \neq \varnothing$. We henceforth assume that (M, g) is four dimensional and globally hyperbolic.

Remark 10: If $x, y \in M$ are causally related in a globally-hyperbolic spacetime (M, g), then there is a causal geodesic segment joining them in view of Proposition 19 in Chapter 14 of [ON83]. This fact has a crucial consequence. If x, y are causally related and both $U \supset J(x, y)$, $U' \supset J(x, y)$ for convex neighbourhoods U, U', then $\sigma_U(x, y) = \sigma_{U'}(x, y)$. Indeed, the unique geodesic segments parametrized on $[0, 1]$ connecting x and y respectively in U and U' must belong to $J(x, y) \subset U \cap U'$ and thus they must coincide. This fact is throughout exploited in [KW91] and provides a well-defined notion of signed squared geodesic distance $\sigma(x, y)$ on the subset of $M \times M$

$$\mathscr{X}_M := \{(x,y) \in M \times M \mid x, y \,\text{causally related}\,,\ J(x,y) \subset U,\ U \,\text{normal convex neighbourhood}\} \,.$$

The definition of Hadamard state according [KW91] passes through the following four steps.

(H1) The so-called (global) **Hadamard parametrix** is defined in [KW91], for every natural n and $\epsilon > 0$, as

$$G^{T,n}_\epsilon(x,y) := \frac{1}{(2\pi)^2}\left[\frac{\Delta(x,y)^{1/2}}{\sigma(x,y) + 2i\epsilon t(x,y) + \epsilon^2} + v^n(x,y)\ln(\sigma(x,y) + 2i\epsilon t(x,y) + \epsilon^2)\right], \quad (x,y) \in \mathscr{O} \,. \tag{8}$$

Above $\mathscr{O} \supset \mathscr{X}_M$ is an open set *supposed to exist* where σ and $G^{T,n}_\epsilon$ are well defined, $t(x, y) := T(x) - T(y)$, where $T : M \to \mathbb{R}$ is global smooth time function[6] increasing towards the future, the branch cut of the logarithm is taken along the negative real axis, and the function $\Delta(x, y)$ and $v_n(x, y)$ are known and defined in terms of $\sigma(x, y)$ and known recursion integrals along the geodesic segment γ_{xy} connecting x and y (see, e.g., Appendix A of [Mo03] and [HM12]).

Remark 11: If $\sigma(x, y)$ and the geodesic segment γ_{xy} connecting x and y are well defined in some neighborhood, then $\Delta(x, y)$ and $v_n(x, y)$ are completely determined in that neighbourhood. This happens in particular for $x, y \in U$ with U normal convex neighbourhood.

(H2) Following [KW91], given a globally hyperbolic spacetime (M, g) with a time orientation and a smooth spacelike Cauchy surface Σ, a **normal neighbourhood** N of Σ is an open set including Σ and such that

(a) $(N, g|_N)$ is a globally hyperbolic spacetime and Σ is a Cauchy surface of it;

[6] I.e., $dT \neq 0$ is everywhere past-directed and $T^{-1}(r)$ is a smooth spacelike Cauchy surface for every $r \in \mathbb{R}$.

(b) $(x, y) \in N \times N$ are causally related in (M, g) iff $(x, y) \in \mathscr{X}_M$.

Lemma 2.2 of [KW91] proves the existence of a normal neighbourhood of any given Cauchy surface Σ.

(H3) Consider an open set $\mathscr{O}' \subset N \times N$ which includes $\mathscr{X}_M \cap (N \times N)$ (i.e., the set of causally related pairs $(x, y) \in N \times N$) and such that its closure in $N \times N$ satisfies $\overline{\mathscr{O}'}^{N\times N} \subset \mathscr{O}$. Finally, $\chi : N \times N \to \mathbb{R}$ is a smooth function such that $\chi(x, y) = 1$ for $(x, y) \in \overline{\mathscr{O}'}^{N\times N}$ and $\chi(x, y) = 0$ for $(x, y) \notin \mathscr{O} \cap (N \times N)$.

(H4) With $\mathscr{O}$, N, T, χ as above, we can state the definition of Hadamard state.

Definition 12: An algebraic state ω on the (Weyl C^* or *) algebra of a real scalar Klein-Gordon field on (M, g) is said to be **globally Hadamard** according to [KW91] if the associated *two-point function*, i.e., a certain bilinear map [KW91] $\Lambda_\omega : C_0^\infty(M)\times C_0^\infty(M) \to \mathbb{C}$, satisfies the following requirement

$$\Lambda_\omega(F_1, F_2) = \lim_{\epsilon\to 0^+} \int_{N\times N} \Lambda_\epsilon^{T,n}(x, y)F_1(x)F_2(y)d\mu_g(x)d\mu_g(y) , \quad \forall F_1, F_2 \in C_0^\infty(N) , \tag{9}$$

where μ_g is the natural measure induced by g on M and

$$\Lambda_\epsilon^{T,n}(x, y) = \chi(x, y)G_\epsilon^{T,n}(x, y) + H^n(x, y) , \tag{10}$$

for every natural n and some associated functions $H^n \in C^n(N \times N)$.

Remark 13: In [KW91], it is proved that Definition 12 is independent of $\mathscr{O}, N, \chi, \Sigma$. Yet, that independence proof assumes at various steps that $G_\epsilon^{T,n}(x, y)$ is well defined, not only on $\mathscr{X}_M$, but also on $\mathscr{O}$ (and $\mathscr{O}'$). In particular, $\sigma(x, y)$ is expected to have the standard behaviour in $\mathscr{O}$: $\sigma(x, y) > 0$ if $x \neq y$ are not causally related. More precisely, $\sigma(x, y)$ is supposed to take the standard form $\sigma(x, x') = -(y^0(x'))^2 + \sum_{\alpha=1}^3 (y^\alpha(x'))^2$ in *Riemannian normal coordinates* y^0, y^1, y^2, y^3 centered at one of the entries (here x) also for non-causally related arguments.

Definition 12 was later proved to be equivalent to a certain *microlocal version* by a famous paper by M. Radzikowski [Rad96a], when assuming the requirement $\Lambda_\omega \in \mathcal{D}'(M \times M)$ (see (2) in Theorem 18 below). This second analytic version (extended to n-dimensional spacetimes with $n \geq 2$) is the one usually nowadays adopted in perturbative aQFT, also including cosmological applications, starting form semiclassical versions of the Einstein equations (see [MPS21] for a recent application). We reccommend [BDFY15] for a recent account on the wide spectrum of applications of Hadamard states (a pedagogical introduction to quasifree Hadamard states and their relevance in aQFT takes place in [KM15] therein). Kay-Wald's version of the Hadamard condition has been later used by R. Verch to prove physically important properties of Hadamard states at algebraic level, like *local quasi equivalence* and *local definiteness* [Ve94]. Using Kay-Wald's definition, Sahlmann and Verch [SV01] extended the formalism to vector-valued quantum fields in a globally hyperbolic spacetime of dimension $n \geq 2$. There, also the equivalent microlocal formulation has been discussed and an extension of the *theorem of propagation of Hadamard singularity* has been established in the fashion of the original formulation [FSW78] of that property of Hadamard states. The use of the Hadamard condition in the study of Hawking radiation can be traced back to [FH90], already before that the precise form of the Hadamard parametrix was stated in [KW91]. Though the microlocal version has been recently employed in applications to aQFT in black-hole background [DMP11, Sa15], the originary [KW91] version of the Hadamard condition have continued to play a crucial role to discuss the Hawking radiation [Wa94], also in terms of a tunneling process [MP12, CMP14] (actually, those works only concern a local version of the Hadamard condition). See in particular the recent interesting work [KPV21] on the Hawking radiation (and partially on the black-hole entropy) for a collapsing black-hole spherically-symmetric spacetime, where the global Hadamard condition has been used.

3.2 A gap in the definition of $G^{T,n}_\epsilon$ and a proposal of solution

The parametrix $G^{T,n}_\epsilon$ is evidently well defined on $\mathscr{X}_M$, but there is no guarantee that it is also well defined on some open neighbourhood $\mathscr{O} \supset \mathscr{X}_M$. Indeed, the open set $\mathscr{O}$ may also contain pairs (p,q) which are *not* causally related and each such pair may be connected by many geodesic segments because Remark 10 does not apply. At this juncture, there is no explicit prescription to smoothly choose a unique geodesic segment for every such pair (p,q) in order to have a well defined $\sigma(p,q)$, which, e.g., satisfies $\sigma(x,y) > 0$ when $x \neq y$ are not causally related. The problem also arises in the definition of $\Delta(p,q)$ and $v^n(p,q)$ as they are computed using a geodesic segment joining p and q as said above.

Instead of attacking the problem directly by trying to establish the existence of a neighbourhood $\mathscr{O} \supset \mathscr{X}_M$ where σ and $G^{T,n}_\epsilon$ are well defined, we adopt a different strategy to circumvent the gap by employing the achievements of Sect 2.2. The strategy relies on minimal modifications of original Kay-Wald's machinery. *For this reason, in author's view, all important results established over the years that rely on Definition 12 (some of them quoted above) are correct.*

Given a four-dimensional globally hyperbolic spacetime (M,g) with a time orientation, choose a strong convex covering $\mathcal{C}$ of M, define the triple $(\mathscr{A},\Gamma,\sigma)$ subordinated to $\mathcal{C}$ as in Theorem 8 and the set

$$\mathscr{X}^{\mathcal{C}}_M := \{(x,y) \in M \times M \mid x, y \,\text{causally related}\,,\ J(x,y) \subset U \in \mathcal{C}\}\,.$$

Notice that $\mathscr{A}$ is an open neighborhood of $\mathscr{X}^{\mathcal{C}}_M$ by construction.

(H1)' Define a (global) **Hadamard parametrix subordinated to** $\mathcal{C}$, for every natural n and $\epsilon > 0$, as

$$G^{T,n,\mathcal{C}}_\epsilon(x,y) := \frac{1}{(2\pi)^2}\left[\frac{\Delta(x,y)^{1/2}}{\sigma(x,y) + 2i\epsilon t(x,y) + \epsilon^2} + v^n(x,y)\ln(\sigma(x,y) + 2i\epsilon t(x,y) + \epsilon^2)\right], \quad (x,y) \in \mathscr{A}\,. \tag{11}$$

Above, $t(x,y) := T(x) - T(y)$, where $T : M \to \mathbb{R}$ is global smooth time function increasing towards the future, the branch cut of the logarithm is taken along the negative real axis, and the functions, σ, Δ and v_n are the ones construted out of $(\mathscr{A},\Gamma,\sigma)$ starting from $\mathcal{C}$.

(H2)' Given a smooth spacelike Cauchy surface Σ of (M,g) (with dimension ≥ 2), a **normal neighbourhood** $N_{\mathcal{C}}$ of Σ **subordinated to** $\mathcal{C}$ is an open set including Σ and such that

(a) $(N, g|_{N_{\mathcal{C}}})$ is a globally hyperbolic spacetime and Σ is a Cauchy surface of it;

(b) $(x,y) \in N_{\mathcal{C}} \times N_{\mathcal{C}}$ are causally related in (M,g) iff $(x,y) \in \mathscr{Z}_M^{\mathcal{C}}$.

Lemma 14: *Given a strong convex covering of M, every smooth spacelike Cauchy surface of (M,g) admits a normal neighbourhood subordinated to $\mathcal{C}$.*

Proof. Use the same proof as the one of Lemma 2.2 of [KW91] with the only difference that all the used normal convex neighbourhoods must be taken in $\mathcal{C}$. □

(H3)' Consider an open set $\mathscr{A}' \subset N_{\mathcal{C}} \times N_{\mathcal{C}}$ which includes $\mathscr{Z}_M^{\mathcal{C}} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$ (i.e., the set of causally related pairs $(x,y) \in N_{\mathcal{C}} \times N_{\mathcal{C}}$) and such that its closure in $N_{\mathcal{C}} \times N_{\mathcal{C}}$ satisfies $\overline{\mathscr{A}'}^{N_{\mathcal{C}} \times N_{\mathcal{C}}} \subset \mathscr{A} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$.

Remark 15: $\mathscr{A}'$ does exist because $\mathscr{Z}_M^{\mathcal{C}} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$ is closed in $N_{\mathcal{C}} \times N_{\mathcal{C}}$ (with the relative topology) and it is included in the open set $\mathscr{A} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$. (The set $\mathscr{Z}_M^{\mathcal{C}} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$ of causally related points in $N_{\mathcal{C}}$ is closed in $N_{\mathcal{C}} \times N_{\mathcal{C}}$ because $N_{\mathcal{C}}$ is globally hyperbolic and Lemma 22 in Chapter 14 of [ON83] is valid[7].)

Finally, taking advantage of the smooth Urysohn lemma, choose a smooth function $\chi : N_{\mathcal{C}} \times N_{\mathcal{C}} \to [0,1]$ such that $\chi(x,y) = 1$ for $(x,y) \in \overline{\mathscr{A}'}^{N_{\mathcal{C}} \times N_{\mathcal{C}}}$ and $\chi(x,y) = 0$ for $(x,y) \notin \mathscr{A} \cap (N_{\mathcal{C}} \times N_{\mathcal{C}})$.

(H4)' With $\mathcal{C}$, $N_{\mathcal{C}}$,T, χ as above, we can give the definition of Hadamard state.

Definition 16: An algebraic state ω on the (Weyl C^* or *) algebra of a real scalar Klein-Gordon field on (M,g) is said to be **globally Hadamard** if the associated *two-point function* $\Lambda_\omega : C_0^\infty(M) \times C_0^\infty(M) \to \mathbb{C}$, satisfies the following requirement

$$\Lambda_\omega(F_1, F_2) = \lim_{\epsilon \to 0^+} \int_{N_{\mathcal{C}} \times N_{\mathcal{C}}} \Lambda_\epsilon^{T,n,\mathcal{C}}(x,y) F_1(x) F_2(y) d\mu_g(x) d\mu_g(y) \,, \quad \forall F_1, F_2 \in C_0^\infty(N_{\mathcal{C}}) \,, \tag{12}$$

where μ_g is the natural measure induced by g on M and

$$\Lambda_\epsilon^{T,n,\mathcal{C}}(x,y) = \chi(x,y) G_\epsilon^{T,n,\mathcal{C}}(x,y) + H^n(x,y) \,, \tag{13}$$

for every natural n and some associated functions $H^n \in C^n(N_{\mathcal{C}} \times N_{\mathcal{C}})$.

Remark 17: An identity is of utmost physical interest: two parametrices subordinated to different strong convex coverings are however identical and also coincide with $G_\epsilon^{T,n}(x,y)$ in (8)

$$G_\epsilon^{T,n,\mathcal{C}}(x,y) = G_\epsilon^{T,n,\mathcal{C}'}(x,y) = G_\epsilon^{T,n}(x,y)$$

when evaluated on causally related points $(x,y) \in N_{\mathcal{C}} \cap N_{\mathcal{C}'}$. In fact, in the said hypothesis, it simultaneously holds $(x,y) \in J(x,y) \subset C \in \mathcal{C}$ and $(x,y) \in J(x,y) \subset C' \in \mathcal{C}'$ and thus, according to Remark 10, the geodesic segments joining x and y in C and C', respectively, coincide. Finally the parametrices coincide as well in view of Remark 11. What happens to $G_\epsilon^{T,n,\mathcal{C}}(x,y)$ for non-causally related points is physically irrelevant and it permits an arbitrary choice of the function χ appearing in $\chi(x,y)G_\epsilon^{T,n,\mathcal{C}}(x,y)$. A change of the function χ can be reabsorbed in a change

[7] The proof appearing in Lemma 3.3 in [Rad96a] of this fact seems to be wrong or incomplete, because σ is not necessarily defined in the target points of the considered sequences, though it grasps the correct insight, with an appropriate use of the *time separation function* τ in place of σ as in Chapter 14 of [ON83].

of the functions H^n. That is a consequence of the fact that, for $x \neq y$ non-causally-related, $\sigma(x,y) > 0$ and no singularity (for $\epsilon \to 0^+$) shows up in the parametrix $G_\epsilon^{T,n,\mathcal{C}}(x,y)$. In other words, the parametrix viewed as a distribution is actually a smooth function for non-causally-related arguments. All that was discussed and clearly emphasized in [KW91] referring to the parametrix $G_\epsilon^{T,n}$. Unfortunately these properties of $G_\epsilon^{T,n}$ rely also on a good behavior of σ in the whole open neighborhood $\mathscr{O}$ (and $\mathscr{O}'$) which is not proved to exist.

3.3 Independence of the choices of $\mathcal{C}, N_\mathcal{C}, T, \chi$ and nice interplay with the microlocal formulation

What remains to be demonstrated is that the given definition of Hadamard state does not depend on the choice of $\mathcal{C}, N_\mathcal{C}, T, \chi$ and that is the corresponding of the microlocal formulation [Rad96a]. In [Rad96a] it was established that a state of a real Klein-Gordon field in a globally hyperbolic spacetime (M,g) is Hadamard in the sense of [KW91] if and only if it satisfies the *microlocal spectral condition* (14) below. (Actually it was done when also assuming the fair hypothesis that the two-point function Λ is a distribution of $\mathcal{D}'(M \times M)$.) As a matter of fact, this result gave rise to an alternative but equivalent definition of Hadamard state.

The presentation of the Hadamard condition in the original sense of [KW91] in [Rad96a, Rad96b] is affected by the issue pointed out above (in the proof of Lemma 3.1 in [Rad96a] in particular) since [Rad96a] includes a faithful summary of relevant ideas and notions appearing in [KW91].

We argue that the statement of Theorem 5.1 in [Rad96a] which establishes the equivalence of the two formulations *is however valid when assuming our definition of Hadamard state according to (H1)'-(H4)'*. Let us re-state here part of Radzikowski's equivalence theorem (excerpt from Theorem 5.1 [Rad96a] with notations adapted to our paper).

Theorem 18: *Let (M,g) be a smooth, time oriented, four-dimensional globally hyperbolic spacetime and $\Lambda \in \mathcal{D}'(M \times M)$, define the Klein-Gordon operator $P := -\Box + m^2 : C^\infty(M) \to C^\infty(M)$ for some real valued $m^2 \in C^\infty(M)$. Choose $\mathcal{C}, T, \chi, N_\mathcal{C}$ as above. Then the following conditions are equivalent.*

(1) Λ *satisfies what follows.*

- *(a) The global Hadamard condition in Definition 16 (referring to the given choice of $\mathcal{C}, T, \chi, N_\mathcal{C}$),*
- *(b) its antisymmetric part is $\frac{i}{2}(\Delta_A - \Delta_R)$ (where $\Delta_{A/R} : C_0^\infty(M) \to C^\infty(M)$ are the advanced/retarded Green operators of P),*
- *(c)* $\Lambda(PF \otimes F') = \Lambda(F \otimes PF') = 0$ *for all real-valued* $F, F' \in C_0^\infty(M)$.

(2) Λ *satisfies what follows.*

- *(a)' The* **microlocal spectral condition**

$$WF(\Lambda) = \{((x_1,k_1),(x_2,k_2) \in T^*M \setminus \mathbf{0} \times T^*M \setminus \mathbf{0} \mid (x_1,k_1),(x_2,-k_2)\,, k_1 \rhd 0\}\,, \quad (14)$$

- *(b) its antisymmetric part is $\frac{i}{2}(\Delta_A - \Delta_R)$,*
- *(c)* $\Lambda(PF \otimes F') = \Lambda(F \otimes PF') = 0$ *for all real-valued* $F, F' \in C_0^\infty(M)$.

((b) and (c) are in particular valid if $\Lambda = \Lambda_\omega$ for an algebraic state ω on the (Weyl C^* or *) algebra of a real Klein Gordon quantum field.)
The thesis is still valid if (b) and (c) in both (1) and (2) are true only mod C^∞.

Sketch of Proof. The proof of of Theorem 5.1 [Rad96a] uses both microlocal analysis arguments and some results from [KW91]. Concerning definitions and facts established in [KW91], it is assumed that (i) the parametrix $G_\epsilon^{T,n}(x,y)$ has the known structure in terms of σ, Δ, v^n in a covering of normal convex neighbourhoods as stated in [KW91], (ii) the Hadamard expansion is well-behaved on the open neighborhood $\mathscr{O}'$ of the causally related points in $N \times N$, where $\sigma(x,y) > 0$ for $x \neq y$ which are not causally related (more precisely it takes the standard form $\sigma(x,x') = -(y^0(x'))^2 + \sum_{\alpha=1}^3 (y^\alpha(x'))^2$ in Riemannian normal coordinates y^0, y^1, y^2, y^3 centered at x), and (iii) the definition of Hadamard state according to [KW91] is independent from the choice of the global time coordinate T. This proof of independence appears in Appendix B of [KW91] and it can be recast without changes for our definition of Hadamard state based on the parametrix $G_\epsilon^{T,n,\mathcal{C}}(x,y)$ and a normal neighborhood $N_\mathcal{C}$. In summary, replacing $\mathscr{O}'$ for $\mathscr{A}'$, using the fact that $G_\epsilon^{T,n,\mathcal{C}}(x,y)$ has the same local structure as $G_\epsilon^{T,n}(x,y)$ in terms of σ and the Hadamard expansion coefficients and *is* well-behaved on $\mathscr{A}$, and exploiting independence of the definition from the choice of T, the proof of Theorem 5.1 [Rad96a] is valid as it stands for Definition 16 of Hadamard state[8]. □

Corollary 19: *The definition of Hadamard state in Definition 12 (based on (H1)'-(H4)' and assuming (b) and (c) of Theorem 18 for Λ_ω) does not depend on the choices of $\mathcal{C}, N_\mathcal{C}, T, \chi$ if $\Lambda_\omega \in \mathcal{D}'(M \times M)$.*

Proof. Item (2) above does not depend on the choices of $\mathcal{C}, N_\mathcal{C}, \chi$. (Independence from the choice of T was independently established in the proof of Theorem 18 using the same proof as in [KW91].) □

To conclude we observe that, following [Rad96b], an algebraic state ω on a Klein-Gordon quantum field on a spacetime (M,g) is said to be **locally Hadamard** if there is a (normal convex) neighborhodd U of every point where, for every natural n, the two-point function of the state Λ_ω can be decomposed as in (10) (i.e., (13)) with Λ_ω in place of $\Lambda^{T,n}$, for $\chi = 1$ and $H^n \in C^n(U \times U)$. It is possible to prove that a state ω such that $\Lambda_\omega \in \mathcal{D}'(M \times M)$ satisfies (b) in Theorem 18 is locally Hadamard in a four-dimensional globally hyperbolic spacetime if and only if it is globally Hadamard. It was established in Theorem 9.2 in [Rad96b] using only the microlocal definition (i.e., (2) in Theorem 18) of Hadamard state and thus that result is valid also with our definition of (global) Hadamard state.

[8] As is known, the proof of Theorem 5.1 in [Rad96a] has a gap. It is the content of the three lines immediately before the proof of (ii) 3 $\Rightarrow$ 2 on p.547. This gap was closed in several independent works, in particular (but not only) [SV01] and [KM15]. In the latter, only the microlocal analysis approach was exploited and thus without relation with the issue with [KW91] definition of Hadamard state. See Remark 23 in [KM15] for a summary on this subject.

Acknowledgments

I am grateful to Franco Cardin, Claudio Dappiaggi, Nicolò Drago, Nicola Pinamonti, and Miguel Sánchez for various discussions over the years on subjects somehow related to the content of this work. I thank Chris Fewster for several technical discussions and suggestions about this paper. I am finally grateful to C. Fewster, F. Kurpicz, N. Pinamonti, and R. Verch who, directly or indirectly, encouraged the author to write down this quite technical note.

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