A geometric perspective on Algebraic Quantum Field Theory

Vincenzo Morinelli
Dipartimento di Matematica, Università di Roma "Tor Vergata"
morinell@mat.uniroma2.it

Dedicated to Karl-Hermann Neeb on the occasion of his sixtieth birthday.

Abstract

In this paper we give a streamlined overview of some of the recent constructions provided with K.-H. Neeb, G. Ólafsson and collaborators for a new geometric approach to Algebraic Quantum Field Theory (AQFT). Motivations, fundamental concepts and some of the relevant results about the abstract structure of these models are here presented.

Contents

1 Introduction

These notes aim to provide a structured guideline, building upon recent results developed in collaboration with K.-H. Neeb and G. Ólafsson, to explore a geometric perspective on Algebraic Quantum Field Theory. In this presentation, we will focus on some selected results from the literature of recent years, illustrating them through specific cases. We hope this approach will assist readers in navigating the main findings and the research directions, while maintaining a balanced view of both the geometric and operator-algebraic perspectives on the related topics.

We start our discussion with some motivations. Algebraic Quantum Field Theory (AQFT) is an axiomatic framework for Quantum Field Theory where a model is specified by a map associating to any open region of the spacetime its von Neumann algebra of local observables, acting on a fixed complex Hilbert space $\mathcal{H}$ (the state space), satisfying fundamental quantum and relativistic assumptions as isotony, locality, Poincaré covariance, positivity of the energy and cyclicity of the vacuum vector for local algebras [Ha96]. In this picture a rich interplay between the geometry and the algebraic parts of the theory appears.

The algebraic structure of the AQFT models, through Tomita–Takesaki theory, contains a large part of the information on the geometry of the model. The Bisognano-Wichmann property and the PCT theorem describe a *geometric action* of the Tomita modular group and the Tomita modular conjugation of local algebras of some specific regions with respect to the vacuum state. They have been verified for a large number of models (see e.g. [BW75, BGL93, Mu01, Mo18, DM20]) and this ensures a very strong relation between the geometry and the algebraic structure of the models that enriches in many directions in the Algebraic approach to Quantum Field Theory. As one of the examples of this interplay we point out that recently operator algebraic methods have proven to be highly effective in studying energy conditions of AQFT models. Here the modular Hamiltonian, namely the logarithm of the Tomita modular operator associated with the local algebra of a specific wedge region with respect to a specific state, plays a central role, see for instance [MTW22, Lo20, CLRR22, CLR20, Ara76, LX18, LM23, LM24, Wi18]. Indeed, by the Bisognano–Wichmann Theorem it has a geometrical meaning and a rigorous analysis from both the algebraic and the geometric perspectives is possible.

Consider an AQFT on Minkowski spacetime. A Rindler wedge and its Poincaré transforms (wedge regions) are fundamental localization regions for von Neumann algebras or particle states and are in 1-1 correspondence with a one–parameter group of boost symmetries (properly parametrized) that fix them as a subset of Minkowski spacetime. The algebraic canonical construction of the free field provided by Brunetti–Guido–Longo (BGL) builds on the wedge-boost correspondence, the Bisognano–Wichmann (BW) property and the PCT Theorem (or the geometric action of the modular conjugation, here called modular reflection), cf. [BGL02]. This approach is actually very general and this is already visible in the BGL paper. Indeed the authors applied this construction to many of the free models of physical interests that have a 1-1 correspondence between specific regions of spacetime and one–parameter subgroups of the symmetry group: In free (possibly conformal) theories on Minkowksi spacetime, on de Sitter space and on the chiral circle.

What we learn is that we can associate bijectively wedge regions of the spacetime to one–parameter subgroups of the Lie group, hence to generators in the Lie algebra. This is the starting point of our analysis. The core of this analysis relies on the understanding of a deep connection between the geometry of real subspaces (the language for localized particle states), given by the Tomita modular operator and modular conjugation, and the geometry of specific elements in the Lie algebra of a Lie group G, called Euler elements. We call an element x of the finite dimensional real Lie algebra $\mathfrak{g}$ an *Euler element* if $\operatorname{ad} x$ is diagonalizable with $\operatorname{Spec}(\operatorname{ad} x) \subseteq \{-1, 0, 1\}$, hence $\operatorname{ad} x$ defines a 3-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_1(x) \oplus \mathfrak{g}_0(x) \oplus \mathfrak{g}_{-1}(x), \qquad \text{where} \qquad \mathfrak{g}_\nu(x) = \ker(\operatorname{ad} x - \nu \operatorname{id}_{\mathfrak{g}})$$

(see [BN04, Ka00] for more details on Euler elements and three graded Lie algebras). The combination of the language of standard subspaces and Euler elements is useful for several reasons. In particular it provides a general structure that matches the geometry of the models with Tomita modular theory and it captures the fundamental localization properties of particle states and von Neumann algebras of observables. This allows to define an abstract framework that properly generalizes the set of AQFT models.

We start this discussion by recalling some fundamental models in Algebraic Quantum Field Theory and introducing our geometric perspective. Once the correspondence between wedges and boosts is clear, we refer it to the notion of abstract wedges, their properties and a classification of simple real Lie algebras supporting such abstract wedges. Then we will see how models in AQFT can be generalized when abstract wedges are taken into account. This first part mainly refers to [MN21]. The fundamental models we are discussing and the related axiomatic framework sits on a causal homogeneous space [FNÓ23, MNO23a, MNO23b, NÓØ21, NÓ21, NÓ22]. Then we pass to discuss a recent construction of non-modular covariant net of standard subspaces, cf. [MN22, Sect. 3.1]. In Sect. 3 we introduce the Euler Element Theorem that ensures that it is possible to deduce our Lie theoretical description of wedges from the Bisognano–Wichmann property and a regularity property. The latter property is strongly related with a localization property, see Sects. 3.2 and 3.3 and [MN24, Sect. 4]. Lastly we show that the von Neumann algebras appearing in this picture are of type III_1 (the Connes classification), see Sect 3.4 and [MN24, Sect. 5]. Many of these results are formulated for general von Neumann algebras and/or standard subspaces in some specific relative position. We explain how they apply to nets of von Neumann algebras (resp. standard subspaces). This approach provides feedback for representation theory (as new constructions of Lie group representations as in [FNÓ23, NÓ21]) and for the algebraic approach to Quantum Field Theory without restrictions to second quantization models. This paper will mainly deal with abstract wedges, a more detailed overview on the properties of wedge regions in causal symmetric space is postponed.

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Notation

- The neutral element of a group G is denoted e, and G_e is the identity component.
- The Lie algebra of a Lie group G is denoted $\mathbf{L}(G)$ or $\mathfrak{g}$.
- For an involutive automorphism σ of G, we write $G^\sigma = \{g \in G \colon \sigma(g) = g\}$ for the subgroup of fixed points and $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension.
- $\mathrm{AU}(\mathcal{H})$ is the group of unitary or antiunitary operators on a complex Hilbert space.
- An (anti-)unitary representation of G_σ is a homomorphism $U \colon G_\sigma \to \mathrm{AU}(\mathcal{H})$ with $U(G) \subseteq \mathrm{U}(\mathcal{H})$ for which $J := U(\sigma)$ is antiunitary, i.e., a conjugation.
- Unitary or (anti-)unitary representations on the complex Hilbert space $\mathcal{H}$ are denoted as pairs $(U, \mathcal{H})$. For a unitary representation $(U, \mathcal{H})$ of G we write: $\partial U(x) = \frac{d}{dt}\big|_{t=0} U(\exp tx)$ for the infinitesimal generator of the unitary one–parameter group $(U(\exp tx))_{t\in\mathbb{R}}$ in the sense of Stone's Theorem.
- If G is a group acting on a set M and $W \subseteq M$ a subset, then the stabilizer subgroup of W in G is denoted $G_W := \{g \in G \colon g.W = W\}$, and $S_W := \{g \in G \colon g.W \subseteq W\}$.
- An element h of a Lie algebra $\mathfrak{g}$ is called

  - *hyperbolic* if $\operatorname{ad} h$ is diagonalizable over $\mathbb{R}$
  - *elliptic* or *compact* if $\operatorname{ad} h$ is semisimple with purely imaginary spectrum.

- Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, then the commutant of $\mathcal{A}$ is $\mathcal{A}' = \{b \in \mathcal{B}(\mathcal{H}) : ab = ba, \forall a \in \mathcal{A}\}$.
  A (complex) $*$-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$. A vector $\Omega \in \mathcal{H}$ is cyclic for $\mathcal{A}$ if $\overline{\mathcal{A}\Omega} = \mathcal{H}$ and it is separating if $\overline{\mathcal{A}'\Omega} = \mathcal{H}$.

2 Preliminaries

2.1 Wedge regions in spactime

In this section we introduce the wedge regions in some of the spacetimes that are relevant for physics and state the correspondence between these regions, specific subgroups of the symmetry group and Lie algebra elements.

Consider Minkowski spacetime $M = \mathbb{R}^{1,d}$ of space dimension d. Here the metric considered is

$$ds^2 = dx_0^2 - dx_1^2 - \ldots - dx_d^2.$$

Let x^2 be the square of the Minkowski length, one defines *timelike vectors* by $x^2 > 0$, *lightlike vectors or null vectors* by $x^2 = 0$, and *spacelike vectors* by $x^2 < 0$. We shall denote $C = \{x \in M : x^2 \geq 0, x_0 \geq 0\}$ the pointed convex closed cone describing the timelike and lightlike future of the origin. Given a region $\mathcal{O}$, its causal complement is

$$\mathcal{O}' = \{y \in \mathbb{R}^{1,d} : (x-y)^2 < 0, \forall x \in \mathcal{O}\}^{\circ}.$$

$\mathcal{O}'$ represents the set of points that correspond to events that do not have causal dependence form events in the region $\mathcal{O}$. The connected component of the identity of the homogenous symmetry group of M is the proper orthochronous Lorentz group $\mathcal{L}_+^{\uparrow} = \mathrm{SO}(1,d)_e$. We denote by $\mathcal{L}_+$ the proper Lorentz group, namely the group generated by $\mathcal{L}_+^{\uparrow}$ and the space and time reflection

$$j_{W_R}(x) = (-x_0, -x_1, x_2, \ldots, x_d)$$

acting on the first two coordinates, hence $\mathcal{L}_+ = \mathcal{L}_+^{\uparrow} \cup \mathcal{L}_+^{\downarrow}$ where $\mathcal{L}_+^{\downarrow} = j_1 \mathcal{L}_+^{\uparrow}$. The associated inhomogeneous symmetry groups, will be denoted by $\mathcal{P}_+^{\uparrow} = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+^{\uparrow}$ and $\mathcal{P}_+ = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+$, the proper orthochronous Poincaré group and the proper Poincaré group, respectively, and $\mathcal{P}_+ = \mathcal{P}_+^{\uparrow} \cup \mathcal{P}_+^{\downarrow} = \mathcal{P}_+^{\uparrow} \cup j_1 \mathcal{P}_+^{\uparrow}$.

In this framework there are some distinguished regions (connected open subsets) of M that have a deep connection with the symmetry group. A *wedge region* $W \subset \mathbb{R}^{1,d}$ is a Poincaré transform of the standard right wedge W_R:

$$W = gW_R, \quad g \in \mathcal{P}_+^{\uparrow}, \quad \text{where} \quad W_R = \left\{x \in \mathbb{R}^{1+d} : |x_0| < x_1\right\}$$

Now, consider the one–parameter group of boosts contained in $\mathcal{P}_+^{\uparrow}$:

$$\Lambda_{W_R}(t) = \begin{pmatrix} \cosh(t) & \sinh(t) & \mathbf{0} \\ \sinh(t) & \cosh(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{1}$$

All the other Poincaré boosts are determined by the adjoint action $\Lambda_W = g\Lambda_{W_R}g^{-1}$, with $g \in \mathcal{P}_+^\uparrow$. It is not by coincidence that we label the boost with the same index of a wedge region. Indeed $\Lambda_W(t)W = W$ and, once the parametrization is fixed, the 1-1 correspondence

$$W \overset{1-1}{\longleftrightarrow} \Lambda_W.$$

Let $h_W \in \mathbf{L}(\mathcal{P}_+^\uparrow)$ be the generator of $\Lambda_W(t) = \exp(th_W)$. Then there is a 1-1 correspondence between wedge regions and the Lie algebra elements that are generators of boost transformations

$$h_W \overset{1:1}{\longleftrightarrow} \Lambda_W(t) = \exp(th_W) \overset{1:1}{\longleftrightarrow} W.$$

Let $W = gW_R$ and $W' = \{x \in M : (x-y)^2 < 0, y \in W\}^\circ$ be the causal complement wedge of a wedge W, the transformation $j_W := gj_{W_R}g^{-1} \in \mathcal{P}_+^\downarrow$ maps W onto W'. Clearly $j_W = j_{W'}$ and it is important to remark that $j_W = \exp(i\pi h_W)$. We have learnt that *wedge regions can be determined by Lie algebra elements* and that there is a natural transformation sending W to W' that happens to be the analytic extension of the one–parameter group Λ_W. We further remark that $\Lambda_{W'}(t) = \Lambda_W(-t)$, hence $h_{W'} = -h_W$. When conformal symmetries are considered, namely the symmetry group is the group of local diffeomorphisms (defined out of meager sets) which preserve the metric tensor up to a positive function, then all the conformal transforms of a wedge in Minkowski space, such as wedges, double cones, future cones and past cones, give rise to conformal wedges and the previous identification with Lie generators continues to hold, see [BGL02, BGL93]. Here, a future and a past cone are intended to be respectively regions of the form $(C+a)^\circ$ and $(-C+b)^\circ$ for some $a, b \in M$ and a double cone is a region of the form $(C+a)^\circ \cap (-C+b)^\circ$ where $b - a \in C^\circ$.

Consider the $d-$dimensional de Sitter spacetime as the subset

$$\mathrm{dS}^d = \{x \in \mathbb{R}^{1,d} : x^2 = -1\} \subset \mathbb{R}^{1,d},$$

endowed with the metric obtained by restriction of the Minkowski metric. The symmetry group of isometries is the Lorentz group $\mathcal{L}_+^\uparrow$ and its $\mathbb{Z}_2$-graded extension $\mathcal{L}_+$. Wedges are defined as $W^{\mathrm{dS}} := W \cap \mathrm{dS}^d$, where $W = gW_R$ is a Minkowksi wedge, with $g \in \mathcal{L}_+^\uparrow$. The one–parameter groups of boosts are $\Lambda_{W^{\mathrm{dS}}} := g\Lambda_{W_R^{\mathrm{dS}}}g^{-1}$ with $g \in \mathcal{L}_+^\uparrow$, where $\Lambda_{W_R^{\mathrm{dS}}} = \Lambda_{W_R} \in \mathcal{L}_+^\uparrow$ and there is a 1-1 correspondence

$$h_{W^{\mathrm{dS}}} \overset{1:1}{\longleftrightarrow} \Lambda_{W^{\mathrm{dS}}}(t) = \exp(th_{W^{\mathrm{dS}}}) \overset{1:1}{\longleftrightarrow} W^{\mathrm{dS}}$$

where $h_{W^{\mathrm{dS}}} \in \mathbf{L}(\mathcal{L}_+^\uparrow)$ and $\exp(h_{W^{\mathrm{dS}}}t) = \Lambda_{W^{\mathrm{dS}}}(t)$. As in the Poincaré case one can define the causal complement of a wedge region $W^{\mathrm{dS}\prime}$ and the reflection $j_{W^{\mathrm{dS}}}$ which satisfies $j_{W^{\mathrm{dS}}} = \exp(i\pi h_{W^{\mathrm{dS}}})$ and $j_{W^{\mathrm{dS}}}W^{\mathrm{dS}} = W^{\mathrm{dS}\prime}$.

As a last example we consider the chiral circle S^1. The symmetry group is the Möbius group $\mathsf{M\ddot{o}b}$, the group of orientation preserving fractional linear transformations fixing $S^1 \subset \mathbb{C}$. One can pass from this picture to the one-point compactification of the real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ through the Cayley transform $\bar{\mathbb{R}} \ni x \mapsto -\frac{x-i}{x+i} \in S^1$. Here the symmetry group becomes $\mathsf{M\ddot{o}b} \simeq \mathrm{PSL}_2(\mathbb{R})$. On S^1 the notion of wedge regions is attached to connected, open, non-empty, non dense intervals I of the circle. In the real line picture $\bar{\mathbb{R}}$, they correspond to bounded intervals, half-lines or the complement of a bounded interval of $\mathbb{R}$ in $\bar{\mathbb{R}}$. There is a 1-1 correspondence between one–parameter groups of "boost"-dilations in $\mathsf{M\ddot{o}b}$ and "wedge"-intervals in S^1 given by

$$h_I \in \mathbf{L}(\mathsf{M\ddot{o}b}) \overset{1:1}{\longleftrightarrow} \delta_I(t) = \exp(h_I t) \in \mathsf{M\ddot{o}b} \overset{1:1}{\longleftrightarrow} I \subset S^1$$

where $\delta_I(t) = g\delta_{I_0}(t)g^{-1}$ with $g \in \mathsf{M\ddot{o}b}$ and $\delta_{I_0}(t)$ in the real line picture given by $\delta_{I_0}(t)x = e^t x$ for $I_0 = (0, +\infty)$. In this picture the wedge reflection $j_I = \exp(i\pi h_I)$ is sending I to $I' := \mathrm{int}(S^1 \backslash I)$

reversing the point ordering of the interval. Also in this case $\delta_{I'}(t)=\delta_I(-t)$, hence $h_{I'}=-h_I$, and $j_{I'}=j_I$. This example becomes relevant when one considers the two dimensional Minkowksi space with chiral coordinates $(u,v)=(x_0+x_1,x_0-x_1)$ and the conformal group $\mathsf{M\ddot{o}b}\times\mathsf{M\ddot{o}b}$ as the symmetry group, cf. [BGL93, MT19].

We will see now that these examples of this interesting correspondence between wedges and one–parameter groups of boost generators are special cases of a more general picture.

2.2 Euler elements

In all the previous examples any element h_W identifying a wedge region W in the related spacetime M has the property that the adjoint action of the Lie element on the Lie algebra is diagonalizable with eigenvalues $\{-1,0,1\}$, namely $\mathrm{Spec}(\mathrm{ad}\,h_W)\subseteq\{-1,0,1\}$. This is particularly evident for the 2+1 dimensional de Sitter spacetime, where $\mathcal{L}_+=\mathrm{PSL}_2(\mathbb{R})$, and $\mathbf{L}(\mathcal{L}_+^\uparrow)\simeq\mathfrak{sl}_2(\mathbb{R})$ and $h_{W_R^{\mathrm{dS}}}=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ provides the standard 3-grading of $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_{-1}=\mathbb{R}\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus\mathbb{R}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\oplus\mathbb{R}\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

The same holds for the Möbius group $\mathsf{M\ddot{o}b}\simeq\mathrm{PSL}_2(\mathbb{R})$ in the real line picture. On the other hand, there is a difference in the wedge geometry: in de Sitter spacetime, there are no proper wedge inclusions such as in the conformal circle S^1, where wedge inclusions correspond to interval inclusions.

We now introduce the natural general setting, where all these examples naturally fit in and the previous difference about the inclusion properties of wedge regions becomes clear. The primary reference for this is [MN21].

An element h in a finite dimensional real Lie algebra $\mathfrak{g}$ is called an *Euler element* if $\mathrm{ad}\,h$ is non-zero and diagonalizable with $\mathrm{Spec}(\mathrm{ad}\,h)\subseteq\{-1,0,1\}$. In particular the eigenspace decomposition with respect to $\mathrm{ad}\,h$ defines a 3-grading of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{g}_1(h)\oplus\mathfrak{g}_0(h)\oplus\mathfrak{g}_{-1}(h),\qquad\text{where}\qquad\mathfrak{g}_\nu(h)=\ker(\mathrm{ad}\,h-\nu\,\mathrm{id}_{\mathfrak{g}})$$

Then $\tau_h(y_j)=\exp(i\pi\,\mathrm{ad}\,h)(y_j)=(-1)^jy_j$ for $y_j\in\mathfrak{g}_j(h)$ defines an involutive automorphism of $\mathfrak{g}$ called an *Euler involution*.

We denote $\mathcal{E}(\mathfrak{g})$ the set of Euler elements in $\mathfrak{g}$. The orbit of an Euler element h under the group $\mathrm{Inn}(\mathfrak{g})=\langle e^{\mathrm{ad}\,\mathfrak{g}}\rangle$ of *inner automorphisms* is denoted with $\mathcal{O}_h=\mathrm{Inn}(\mathfrak{g})h\subseteq\mathfrak{g}$. We say that h is *symmetric* if $-h\in\mathcal{O}_h$. We remark that symmetric Euler elements are always contained in a subalgebra $\mathfrak{h}\simeq\mathfrak{sl}_2(\mathbb{R})$ of the Lie algebra $\mathfrak{g}$. Non-symmetric Euler elements are contained in a subalgebra $\mathfrak{h}\simeq\mathfrak{gl}_2(\mathbb{R})$ of $\mathfrak{g}$ (cf. [MN22, Lem. 2.15]). Let G be a Lie group, assume that τ is an Euler involution that integrates to an involution on G, then $G_\tau=G\rtimes_\tau\mathbb{Z}_2$ is well-defined and one has a continuous homomorphism $\epsilon\colon G_\tau\to\{\pm1\}$ where

$$G=\epsilon^{-1}(1)\quad\text{ and }\quad G\,\tau=\epsilon^{-1}(-1),$$

so that $G\trianglelefteq G_\tau$ is a normal subgroup of index 2 and $G\tau=G_\tau\setminus G$. Note that if $G=\mathrm{Inn}(\mathfrak{g})$ or G is simply connected, then τ always integrate to an involution on G. For sake of simplicity assume here that that G is connected and G is center free.

The set

$$\mathcal{G}_E:=\mathcal{G}_E(G_\tau):=\{(h,\tau_h)\in\mathfrak{g}\times G\tau\colon\ h\in\mathcal{E}(\mathfrak{g})\}$$

is called the *abstract Euler wedge space* of G_τ and an element $(h, \tau_h) \in \mathcal{G}_E$ is an *Euler couple* or *Euler wedge*. In the case G is not center free or τ is not an Euler involution, Euler wedges are couples $(h, \sigma) \in \mathfrak{g} \times G\tau$ such that $h \in \mathcal{E}(\mathfrak{g})$ and σ is an involution such that $\mathrm{Ad}\,\sigma = \mathrm{Ad}\,\tau_h$. Given a fixed couple $W_0 = (h, \tau_h) \in \mathcal{G}_E$, the orbits

$$\mathcal{W}_+(W_0) := G.W_0 \subseteq \mathcal{G}_E \quad \text{and} \quad \mathcal{W}(W_0) := G_\tau.W_0 \subseteq \mathcal{G}_E$$

are called the *positive* and the *full abstract wedge space containing* W_0.

Covariance of Euler wedges. A G_τ-covariant action on $\mathcal{G}_E$ can be defined as follows. Consider the *twisted adjoint action* of G_{τ_h} which changes the sign on odd group elements:

$$\mathrm{Ad}^\epsilon \colon G_\tau \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}^\epsilon(g) := \epsilon(g)\,\mathrm{Ad}(g). \tag{2}$$

Then G_τ acts on $\mathcal{G}_E$ by

$$g.(h, \tau_h) := (\mathrm{Ad}^\epsilon(g)h, g\tau_h g^{-1}). \tag{3}$$

Order structure on Euler wedges. Given a "positive" cone in the Lie algebra one can define an order structure: Given an $\mathrm{Ad}^\epsilon(G_{\tau_h})$-invariant pointed closed convex cone $C_\mathfrak{g} \subseteq \mathfrak{g}$, one defines the subsets

$$C_\pm := \pm C_\mathfrak{g} \cap \mathfrak{g}_{\pm 1}(h),$$

and the semigroup,

$$S_W := \exp(C_+) G_W \exp(C_-) = G_W \exp\big(C_+ + C_-\big)$$

([Ne22, Thm. 2.16]). Then the partial order on the orbit $G.W \subseteq \mathcal{G}_E$ is defined by

$$g_1.W \leq g_2.W \quad :\Longleftrightarrow \quad g_2^{-1} g_1 \in S_W. \tag{4}$$

In particular, $g.W \leq W$ is equivalent to $g \in S_W$. As an example, consider $G = \mathrm{PSL}_2(\mathbb{R})$ and its $\mathbb{Z}_2$-graded extension G_{τ_h}. Then, if C is chosen to be trivial, we are in the situation where there are no wedge inclusions, hence in the de Sitter wedge geometry. If we choose

$$C := \left\{ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : b \geq 0, c \leq 0, a^2 \leq -bc \right\} \subset \mathfrak{sl}_2(\mathbb{R}) \tag{5}$$

we are in the situation where there are wedge inclusions, hence of the circle S^1 wedge geometry picture. Namely the choice of C influences the geometry of the model.

Locality and duality properties of Euler wedges. The notion of a "causal complement" is defined on the abstract wedge space as follows: For $W = (h, \tau) \in \mathcal{G}_E$, we define the *dual wedge* by $W' := (-h, \tau) = \tau.W$. Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (3). The inclusion property of wedges fits with the duality property in the sense that the expression $W_1 \subset W_2'$ is meaningful. We shall refer to this condition as the *locality property of the Euler couples*. This relation aligns with the geometric interpretation within the context of wedge domains in spacetime manifolds.

Note that it is not always the case that $W_0' \in \mathcal{W}_+(W_0)$. As an example, consider the translation-dilation group G acting on the real line. A wedge is a right half-line $I = \mathbb{R}^+ + a$ and is associated 1-1 to the positive dilations $\delta_I(t)x = e^t(x - a) + a$ where $x \in \mathbb{R}$. The complement interval I' is a negative half-line $I' = \mathbb{R}^- + a$, it has associated the one-parameter group $\delta_{I'}(t) = \delta_I(-t)$. There exists no $g \in G$ such that $gI = I'$ and one needs to include the τ_I to reflect I to I', namely $I' \in \mathcal{W}(I)$ but $I' \notin \mathcal{W}_+(I)$. This correspond to the fact that h is not symmetric in $\mathfrak{g}$, see [MN21]. We further remark that if $Z(G)$ is non trivial, many abstract wedges correspond to an Euler element. This case is discussed in [MN21, Sect. 2.4.2].

We have now established a suitable framework for defining an abstract index set for Quantum Field Theory. Notably, this structure extends beyond well-known examples within the physics context. One can classify the Lie algebras $\mathfrak{g}$ supporting a three-grading, hence Euler wedges. In particular, the Lie group $\mathrm{Inn}(\mathfrak{g})$ and its extension $\mathrm{Inn}(\mathfrak{g})_{\tau_h}$ generalize the fundamental examples outlined in the previous section. If a simple real Lie algebra $\mathfrak{g}$ supports Euler elements then Theorem 3.10 in [MN21] specifies that its restricted root system must be one of the following:

- A_n : n $\mathrm{Inn}(\mathfrak{g})$-orbits of Euler elements
- B_n : one $\mathrm{Inn}(\mathfrak{g})$-orbit of Euler elements
- C_n : one $\mathrm{Inn}(\mathfrak{g})$-orbit of Euler elements
- D_n : three $\mathrm{Inn}(\mathfrak{g})$-orbits of Euler elements
- E_6 : two $\mathrm{Inn}(\mathfrak{g})$-orbits of Euler elements
- E_7 : one $\mathrm{Inn}(\mathfrak{g})$-orbit of Euler elements

Here is the list of three graded Lie algebras:

	$\mathfrak{g}$	$\Sigma(\mathfrak{g},\mathfrak{a})$	h	$\mathfrak{g}_1(h)$
1	$\mathfrak{sl}_n(\mathbb{R})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{R})$
2	$\mathfrak{sl}_n(\mathbb{H})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{H})$
3	$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{C})$
4	$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{R})$
5	$\mathfrak{u}_{n,n}(\mathbb{H})$	C_n	h_n	$\mathrm{Aherm}_n(\mathbb{H})$
6	$\mathfrak{so}_{p,q}(\mathbb{R})$	$B_p\ (p<q),\ D_p\ (p=q)$	h_1	$\mathbb{R}^{p+q-2}$
7	$\mathfrak{so}^*(4n)$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{H})$
8	$\mathfrak{so}_{n,n}(\mathbb{R})$	C_n	h_n	$\mathrm{Alt}_n(\mathbb{R})$
9	$\mathfrak{e}_6(\mathbb{R})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O}_{\mathrm{split}})$
10	$\mathfrak{e}_{6(-26)}$	A_2	h_1	$M_{1,2}(\mathbb{O})$
11	$\mathfrak{e}_7(\mathbb{R})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O}_{\mathrm{split}})$
12	$\mathfrak{e}_{7(-25)}$	C_3	h_3	$\mathrm{Herm}_3(\mathbb{O})$
13	$\mathfrak{sl}_n(\mathbb{C})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{C})$
14	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{C})$
15a	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1	$\mathbb{C}^n$
15b	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1	$\mathbb{C}^n$
16	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n	$\mathrm{Alt}_n(\mathbb{C})$
17	$\mathfrak{e}_6(\mathbb{C})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O})_{\mathbb{C}}$
18	$\mathfrak{e}_7(\mathbb{C})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O})_{\mathbb{C}}$

Table 1: Simple 3-graded Lie algebras. We follow the conventions of the tables in [Bo90a] for the classification of irreducible root systems, the enumeration of the simple roots and the associated Euler elements, cf. [MN21]

One can further classify the orbits of symmetric Euler elements, leading to a reduced classification as follows:

- A_{2n-1} : one $\mathrm{Inn}(\mathfrak{g})$-orbit of symmetric Euler elements
- B_n : one $\mathrm{Inn}(\mathfrak{g})$-orbit of symmetric Euler elements

- C_n : one $\mathrm{Inn}(\mathfrak{g})$-orbit of symmetric Euler elements
- D_{2n+1} : one $\mathrm{Inn}(\mathfrak{g})$-orbit of symmetric Euler elements
- D_{2n} : two $\mathrm{Inn}(\mathfrak{g})$-orbits of symmetric Euler elements
- E_7 : one $\mathrm{Inn}(\mathfrak{g})$-orbit of symmetric Euler elements

Here is the list of simple hermitian Lie algebras $\mathfrak{g}^\circ$ supporting Euler elements. Theorem [MN21, Prop. 3.11] ensures that they have to be of tube type, namely their restricted root system is of type C_r.

$\mathfrak{g}^\circ$ (hermitian)	$\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$	$\mathfrak{g} = (\mathfrak{g}^\circ)_\mathbb{C}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	symm. Euler element h
$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	$\mathfrak{sl}_{2n}(\mathbb{C})$	A_{2n-1}	h_n
$\mathfrak{so}_{2,2n-1}(\mathbb{R}), n > 1$	C_2	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}_{2,2n-2}(\mathbb{R}), n > 2$	C_2	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1
$\mathfrak{so}^*(4n)$	C_n	$\mathfrak{so}_{4n}(\mathbb{C})$	D_{2n}	h_{2n-1}, h_{2n}
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Table 2: Simple hermitian Lie algebras $\mathfrak{g}^\circ$ of tube type

2.3 Wedge domains in causal homogeneous spaces

Let G be a connected Lie group, H a closed subgroup. The associated homogeneous space is given by the quotient $M = G/H$. It is said to be *causal* if there exists a G-invariant field of pointed generating closed convex cones $\{C_m\}_{m\in M}$ with $C_m \subset T_m(M)$ the tangent space at the point $m \in M$. The Minkowski space, as well as the de Sitter space, is then a causal homogeneous space, because of the causal structure given by the Minkowksi metric.

Given a causal manifold M and an Euler element $h \in \mathfrak{g}$, the *modular vector field* is

$$X_h^M(m) := \frac{d}{dt}\Big|_{t=0} \exp(th).m. \tag{6}$$

Motivated by the Bisognano–Wichmann property (BW) and its consequences in AQFT, the modular flow should be timelike future-oriented. It corresponds to the inner time evolution of Rindler wedges (see [CR94] and also [BB99, BMS01], [CLRR22, §3]). Then the connected component

$$W := W_M^+(h)_{eH} \tag{7}$$

of the base point $eH \in M$ in the *positivity region*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\} \tag{8}$$

is the natural candidate for a wedge domain if $eH \in W_M^+(h)$ (or its boundary). As a concrete example, in Minkowski and de Sitter spacetimes the wedge W_Λ is determined as the set of points $x \in M$ such that the vector field at x associated to the one-parameter group Λ of boosts is timelike and future pointing.

In this context we shall consider causal symmetric spaces that are homogeneous space of the form $M = G/H$, where G is a connected Lie group, and there exists an involution τ on G for which $H \subset G^\tau$ is a union of connected components. We can assume G to be simple and center free. Let $C \subset \mathfrak{g}^{-\tau} \simeq T_{eH}(M)$ be a pointed generating closed H-invariant convex cone producing the causal structure on M by G-action. We can distinguish symmetric spaces that are *compactly causal*, namely the cone C° contains only elliptic elements and *non-compactly causal symmetric spaces*, namely the cone C° contains only hyperbolic elements (see [HÓ96]). The Anti-de Sitter spacetime belongs to the first family and de Sitter spacetime belongs to second one. The wedge regions, as introduced above, have been studied for compactly and non-compactly causal symmetric spaces in [NÓ23] and [NÓ22, MNO23a, MNO23b], respectively.

Assume that the center of G is trivial, then there is a 1-1 correspondence between Euler elements and abstract Euler wedges (Euler couples). In particular the stabilizers $G^{(h,\tau_h)}$ and G^h coincide and one can identify a wedge orbit $\mathcal{W}_+(h,\tau_h) \subseteq \mathcal{G}_E$ with the adjoint orbit $\mathcal{O}_h = \mathrm{Ad}(G)h$. We thus obtain a natural map from $\mathcal{W}_+(h,\tau_h) \cong \mathcal{O}_h$ to regions in M by $g.(h,\tau_h) \mapsto g.W_M^+(h)$. If, in addition, G^h preserves the connected component $W \subseteq W_M^+(h)$, as it happens if $W_M^+(h)$ is connected, one can define a map from the abstract wedge space $\mathcal{W}_+(h,\tau_h) \ni W = (k,\tau_k)$ to the geometric wedge regions $W_M^+(h)$ on M.

Given a real connected simple Lie group G such that the Lie algebra $\mathfrak{g}$ contains an Euler element $h \in \mathcal{E}(\mathfrak{g})$, one can construct a non-compactly causal symmetric space as $M = G/H$ where H is an open subgroup of G^τ, the centralizer of the involution $\tau = \tau_h\theta$ with θ Cartan involution of $\mathfrak{g}$ such that $\theta(h) = -h$ that integrates on G. Note that within this picture $h \in T_{\mathbf{e}}(M) \simeq \mathfrak{g}^{-\tau}$ where $\mathbf{e} = eH$ is the basepoint of M. Furthermore one can define a G-invariant field of pointed generating closed convex cones $C_{\mathbf{p}} \subset T_{\mathbf{p}}(M)$ such that $h \in C_{\mathbf{e}}^\circ$, hence $\mathbf{e} \in W_M^+(h)$, see [MNO23a]. In this way one can associate a wedge domain $W(h) \subset M$ to any $h \in \mathcal{E}(\mathfrak{g})$. A classification result of the local structure of such non compactly-causal homogeneous spaces and connectedness of $W_M^+(h)$ is described in [MNO23a, Thm. 4.21], [MNO23b, Thm. 7.1].

As an example, consider the de Sitter space dS of dimension d and its symmetry group, the Lorentz group, $G = \mathrm{SO}(1,d)_e$. Let $h := h_{W_R^{\mathrm{dS}}}$ be the generator of the one-parameter group of boosts in (1). Then following [MNO23a], one can consider the involution $\tau = \tau_h\theta$ on the Lie algebra $\mathfrak{g} = \mathfrak{so}(1,d)$ where τ_h is the Euler involution associated to the Euler element h and θ is the Cartan involution of the Lie algebra $\mathfrak{g}$ promoted to G. In particular

$$\theta = \mathrm{Ad}\left(\mathrm{diag}(-1, \mathbf{1}_d)\right), \quad \tau_h = \mathrm{Ad}\left(\mathrm{diag}(-1,-1,\mathbf{1}_{d-1})\right), \quad \tau = \mathrm{Ad}\left(\mathrm{diag}(1,-1,\mathbf{1}_{d-1})\right).$$

Let $\mathbf{e}_1 = (0,1,0,\ldots,0) \in \mathrm{dS} \subset \mathbb{R}^{1,d}$ and $G_{\mathbf{e}_1}$ be its stabilizer group, then the centralizer of τ in G is described explicitly by

$$G^\tau \simeq G_{\mathbf{e}_1} \rtimes \{1, r_{1,2}(\pi)\} \simeq G_{\mathbf{e}_1} \rtimes \mathbb{Z}_2$$

where $r_{1,2}(\pi)$ is the π-rotation in the first two spatial coordinates. In particular G^τ has two connected components and $G_{\mathbf{e}_1}$ is the identity component fixing $\mathbf{e}_1$. Then de Sitter space dS is isomorphic to $G/G_{\mathbf{e}_1}$ which is a causal symmetric space but the quotient G/G^τ is the projective de Sitter space which is not a causal space: a global time orientation time orientation is missing. With the identification $\mathrm{dS} \simeq G/G_{\mathbf{e}_1}$ one has that $W_M^+(h) = W_R^{\mathrm{dS}}$. We remark that what happens in this example is a general fact, namely when h is a *symmetric* Euler element, then $M = G/G^\tau$ is never a causal manifold [MNO23a, Thm. 4.19].

2.4 The geometry of nets of real subspaces

In this section we recall some fundamental properties of the geometry of standard subspaces and one-particle nets. We refer to [LRT78, BGL02, MN21, NÓ17, Lo08] for more details. Particles in Quantum Field Theory are defined better through their symmetries rather than pointwise properties and their meaning is clear when the symmetry group is large enough. For instance, one-particle states of a free Quantum Field Theory on Minkowski spacetime are unit vectors of the Hilbert space supporting a unitary irreducible positive energy representation of the Poincaré group.

The free field is the most fundamental model in AQFT. We will see how the language of the standard subspaces captures the right features of the free model to allow a generalization of the AQFT on abstract sets of wedges and on causal symmetric spaces.

2.4.1 Scalar free fields in Algebraic Quantum Field Theory

The free scalar mass $m \geq 0$ Quantum Field Theory on Minkowksi spacetime can be constructed as follows. Let U be the (anti)-unitary positive energy representation of the proper Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathrm{SO}(1,3)_e$. We can assume that the Hilbert space is $\mathcal{H} = L^2(\Omega_m, d\Omega_m)$, where $\Omega_m = \{p \in M : p^2 = m^2, p_0 > 0\}$ and $d\Omega_m$ is the Lorentz invariant measure on Ω_m. Let $f \in \mathcal{S}(M)$, then the functions $\widehat{f}|_{\Omega_m}$ define a complex linear dense subset of $\mathcal{H}$, where $\widehat{f}(p) = \frac{1}{(2\pi)^2}\int_M e^{-ipx} f(x)dx$ is the Fourier transform of f. On this Hilbert space the unitary scalar representation acts by

$$(U(g)f)(x) = f(g^{-1}x), \qquad g \in \mathcal{P}_+^\uparrow.$$

This representation extends by the same formula to a (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+$, [Va85, Thm. 9.10]. We can define the real subspace

$$\mathsf{H}(\mathcal{O}) = \overline{\{\widehat{f}|_{\Omega_m} : f \in C_0^\infty(M), \operatorname{supp} f \subset \mathcal{O}\}}$$

with $\mathcal{O} \subset M$ open bounded region. This is the set of one-particle states localized in $\mathcal{O}$. Note that the following properties hold

- Isotony: if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathsf{H}(\mathcal{O}_1) \subset \mathsf{H}(\mathcal{O}_2)$,
- Locality: if $\mathcal{O}_1 \subset \mathcal{O}_2'$, then $\mathsf{H}(\mathcal{O}_1) \subset \mathsf{H}(\mathcal{O}_2)'$, where $\mathsf{H}(\mathcal{O}_2)'$ is the symplectic complement of $\mathsf{H}(\mathcal{O}_2)$ defined by $\mathsf{H}(\mathcal{O}_2)' := \{\xi \in \mathcal{H} : \ \mathrm{Im}\langle \xi, \eta\rangle = 0, \forall \eta \in \mathsf{H}(\mathcal{O}_2)\}$,
- Covariance: if $g \in \mathcal{P}_+$, then $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(gO)$,
- Reeh–Schlieder: $\overline{\mathsf{H}(\mathcal{O}) + i\mathsf{H}(\mathcal{O})} = \mathcal{H}$ for all $\mathcal{O} \subset M$.

Wedge subspaces can be defined as

$$\mathsf{H}(W) = \overline{\sum_{\mathcal{O} \subset W} \mathsf{H}(\mathcal{O})} \tag{9}$$

The free field is the second quantization net of von Neumann algebras on the Fock space. It is constructed as follows. Given the one-particle Hilbert space $\mathcal{H}$, we can define the Fock space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes_s^n} \oplus \mathbb{C} \cdot \Omega,$$

where $\mathcal{H}^{\otimes_s^n}$ is the symmetrized n-fold tensor product of the one-particle Hilbert space $\mathcal{H}$.

For every vector $\xi \in \mathcal{H}$ the associated Weyl unitary $\mathrm{w}(\xi)$ on the Fock space is defined by the commutation relations

$$\mathrm{w}(\xi)\mathrm{w}(\eta) = e^{-\frac{1}{2}\mathrm{Im}(\xi,\eta)}\mathrm{w}(\xi + \eta), \qquad \xi, \eta \in \mathcal{H} \tag{10}$$

and the expectation value on the vacuum state

$$(\Omega, \mathrm{w}(\xi)\Omega) = e^{-\frac{1}{4}\|\xi\|^2}. \tag{11}$$

Note that the map $\mathcal{H} \ni h \mapsto \mathrm{w}(h) \in \mathcal{F}(\mathcal{H})$ is strongly continuous. Given a real subspace $H \subset \mathcal{H}$, we can associate a the von Neumann algebra acting on the Fock space as follows

$$R(H) := \{\mathrm{w}(\xi) : \xi \in H\}'' \subseteq \mathcal{F}(\mathcal{H}) \ , \tag{12}$$

Let H and H_a be closed, real linear subspaces of $\mathcal{H}$. Then we have

(*a*) $R(\sum_a H_a) = \bigvee_a R(H_a)$,

(*b*) $R(\cap_a H_a) = \bigcap_a R(H_a)$,

(*c*) $R(H)' = R(H')$,

where $\bigvee$ denotes the von Neumann algebra generated (cf. [Ara63, LRT78, LMR16]). We can now define the free scalar net of von Neumann algebras as

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) := R(\mathsf{H}(\mathcal{O})) \subseteq \mathcal{F}(\mathcal{H}).$$

It satisfies

- **Isotony:** if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}_U(\mathcal{O}_2)$
- **Locality:** if $\mathcal{O}_1 \subset \mathcal{O}_2'$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$
- **Poincaré covariance and positivity of the energy**: there exists a (anti-)unitary representation $\widetilde{U}$ of $\mathcal{P}_+$ on $\mathcal{F}(\mathcal{H})$ such that $\widetilde{U}(g)\Omega = \Omega$ and for $g \in \mathcal{P}_+$, we have $\widetilde{U}(g)\mathcal{A}(\mathcal{O})\widetilde{U}(g)^* = \mathcal{A}(gO)$. Furthermore the joint spectrum of the tranlsation generators is contained in the forward lightcone C.
- **Reeh–Schlieder:** $\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{F}(\mathcal{H})$ for any open $\mathcal{O} \subset M$.

We recall that, given a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ with a cyclic and separating vector $\Omega \in \mathcal{H}$, one can define the Tomita operator $S_{\mathcal{A},\Omega}$ as the closure of the anti-linear involution

$$\mathcal{A}\Omega \ni a\Omega \longmapsto a^*\Omega \in \mathcal{A}\Omega.$$

The polar decomposition $S_{\mathcal{A},\Omega} = J_{\mathcal{A},\Omega}\Delta_{\mathcal{A},\Omega}^{1/2}$ defines the self-adjoint **modular operator** Δ and the anti-unitary **modular conjugation** $J_{\mathcal{A},\Omega}$ that satisfy $J_{\mathcal{A},\Omega}\Delta_{\mathcal{A},\Omega}J_{\mathcal{A},\Omega} = \Delta_{\mathcal{A},\Omega}^{-1}$.

The free scalar net $\mathcal{A}(\mathcal{O})$ satisfies the **Bisognano–Wichmann property**, that claims

$$\widetilde{U}(\exp(2\pi t h_R)) = \Delta_{\mathcal{A}(W_R),\Omega}^{-it}, \qquad t \in \mathbb{R}$$

and the **modular reflection property**

$$\widetilde{U}(\tau_{h_R}) = J_{\mathcal{A}(W_R),\Omega}.$$

We remark that this is weaker than the PCT Theorem. The latter ensures an antiunitary implementation of the space, time and charge reflection on the Hilbert space (not only of the wedge coordinates reflection). In even spacetime dimension, the space and time reflection differs from τ_{W_R} by π-rotations, hence the two statements are equivalent, see [BY01].

2.4.2 Nets of standard subspaces

The mathematical structure of the real local subspaces is recalled in this subsection. We call a closed real subspace H of the complex Hilbert space $\mathcal{H}$ cyclic if $\mathsf{H}+i\mathsf{H}$ is dense in $\mathcal{H}$, separating if $\mathsf{H}\cap i\mathsf{H}=\{0\}$, and *standard* if it is cyclic and separating. We write $\mathrm{Stand}(\mathcal{H})$ for the set of standard subspaces of $\mathcal{H}$. The symplectic complement (or symplectic orthogonal) of a real subspace H is defined by the symplectic form $\mathrm{Im}\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ via

$$\mathsf{H}'=\{\xi\in\mathcal{H}:\mathrm{Im}\langle\xi,\eta\rangle=0,\forall\eta\in\mathsf{H}\}=(i\mathsf{H})^{\perp_{\mathrm{Re}\langle\cdot,\cdot\rangle}},$$

where $\mathrm{Re}\langle\cdot\,,\cdot\rangle$ is the real part of the scalar product of $\mathcal{H}$. Then H is separating if and only if H' is cyclic, hence H is standard if and only if H' is standard. For a standard subspace H, one can define the *Tomita operator* S_{H} as the closed antilinear involution

$$S_{\mathsf{H}}:\mathsf{H}+i\mathsf{H}\ni\xi+i\eta\longmapsto\xi-i\eta\in\mathsf{H}+i\mathsf{H}.$$

The polar decomposition $J_{\mathsf{H}}\Delta_{\mathsf{H}}^{\frac{1}{2}}$ of this operator defines an antiunitary involution J_{H} (a conjugation) and the modular self-adjoiont operator Δ_{H}. We then have

$$J_{\mathsf{H}}\mathsf{H}=\mathsf{H}',\quad \Delta_{\mathsf{H}}^{it}\mathsf{H}=\mathsf{H}\qquad\text{for every}\quad t\in\mathbb{R}$$

and the modular relations

$$J_{\mathsf{H}}\Delta_{\mathsf{H}}^{it}J_{\mathsf{H}}=\Delta_{\mathsf{H}}^{it}\qquad\text{for every}\quad t\in\mathbb{R}.\tag{13}$$

We remark that given a von Neumann algebra $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$ with a cyclic and separating vector $\Omega\in\mathcal{H}$ then one can define the standard subspace $\mathsf{H}_{\mathcal{A}}=\overline{\mathcal{A}_{\mathrm{sa}}\Omega}$, where $\mathcal{A}_{\mathrm{sa}}=\{a\in\mathcal{A}:a=a^*\}$. It is easy to see that $S_{\mathsf{H}_{\mathcal{A}}}=S_{\mathcal{A},\Omega}$, namely the standard subspace $\mathsf{H}_{\mathcal{A}}$ contains all the information about the Tomita operator of the von Neumann algebra $\mathcal{A}$ with respect to the vector state Ω.

One can completely reconstruct the standard subspace from the Tomita operators as

$$\mathsf{H}=\mathrm{Fix}(J_{\mathsf{H}}\Delta_{\mathsf{H}}^{\frac{1}{2}})=\ker(1-J_{\mathsf{H}}\Delta_{\mathsf{H}}^{\frac{1}{2}})\tag{14}$$

([Lo08, Thm. 3.4]). This construction leads to a one-to-one correspondence between couples (Δ,J) satisfying the modular relation:

$$J\Delta^{it}J=\Delta^{it},\qquad t\in\mathbb{R}\tag{15}$$

and standard subspaces $\mathsf{H}=\mathrm{Fix}(J\Delta^{\frac{1}{2}})$, see [Lo08, Prop. 3.2].

Standard subspaces have a natural structure that adapts to the language of AQFT as one can see here

- **Covariance property**:([Mo18, Lem. 2.2]). Let $\mathsf{H}\subset\mathcal{H}$ be a standard subspace and $U\in\mathrm{AU}(\mathcal{H})$ be a unitary or anti-unitary operator. Then $U\mathsf{H}$ is also standard and $U\Delta_{\mathsf{H}}U^*=\Delta_{U\mathsf{H}}^{\epsilon(U)}$ and $UJ_{\mathsf{H}}U^*=J_{U\mathsf{H}}$, where $\epsilon(U)=1$ if U is unitary and $\epsilon(U)=-1$ if it is anti-unitary.

- **Duality property**([Lo08, Prop. 2.13]): Let $\mathsf{H}\subset\mathcal{H}$ be a standard subspace, and (Δ,J) the associated operators, then the symplectic complement H' is associated to the couple (Δ^{-1},J).

- **Inclusion property** ([Lo08, Thms. 3.15, 3.17], [BGL02, Thm. 3.2]): The results we indicate here relate an inclusion property of standard subspaces and positivity of the generator of a related one-parameter group.

  Let $\mathsf{H}\subset\mathcal{H}$ be a standard subspace and $U(t)=e^{itP}$ be a unitary one-parameter group on $\mathcal{H}$ with generator P.

(a) If $\pm P > 0$ and $U(t)\mathsf{H} \subset \mathsf{H}$ for all $t \geq 0$, then

$$\Delta_{\mathsf{H}}^{-is/2\pi} U(t) \Delta_{\mathsf{H}}^{is/2\pi} = U(e^{\pm s} t) \quad \text{and} \quad J_{\mathsf{H}} U(t) J_{\mathsf{H}} = U(-t) \quad \text{for all} \quad t, s \in \mathbb{R}. \quad (16)$$

(b) If $\Delta_{\mathsf{H}}^{-is/2\pi} U(t) \Delta_{\mathsf{H}}^{is/2\pi} = U(e^{\pm s} t)$ for $s, t \in \mathbb{R}$, then the following are equivalent:

(1) $U(t)\mathsf{H} \subset \mathsf{H}$ for $t \geq 0$;

(2) $\pm P$ is positive.

The first part of this result is the standard subspace version of the Borchers Theorem and the second part is its converse.

We can now present an axiomatic framework for nets of standard subspaces on abstract wedges

Definition 2.1. Let G be a connected Lie group with trivial center, such that $\mathcal{E}(\mathfrak{g}) \neq \{0\}$. We fix an Euler element and $h \in \mathfrak{g}$ and assume that τ_h integrates to an involution on G and let $W_0 = (h, \tau_h)$. Let C be an $\mathrm{Ad}^\epsilon(G_{\tau_h})$-invariant pointed convex cone in the Lie algebra $\mathfrak{g}$ of G. Let $(U, \mathcal{H})$ be a unitary representation of G and

$$\mathsf{N} \colon \mathcal{W}_+ := \mathcal{W}(W_0) \to \mathrm{Stand}(\mathcal{H}) \quad (17)$$

be a map, also called a *net of standard subspaces.* We consider the following properties:

(HK1) **Isotony:** $\mathsf{N}(W_1) \subseteq \mathsf{N}(W_2)$ for $W_1 \leq W_2$, , $W_1, W_2 \in \mathcal{W}_+$.

(HK2) **Covariance:** $\mathsf{N}(gW) = U(g)\mathsf{N}(W)$ for $g \in G$, $W \in \mathcal{W}_+$.

(HK3) **Spectral condition:** $C \subseteq C_U := \{x \in \mathfrak{g} \colon -i\partial U(x) \geq 0\}$. We then say that U is *C-positive.* Note that C_U is pointed if and only if $\ker(U)$ is discrete.

(HK4) **Locality:** If $W \in \mathcal{W}_+$ is such that $W' \in \mathcal{W}_+$, then $\mathsf{N}(W') \subset \mathsf{N}(W)'$.

(HK5) **Bisognano–Wichmann (BW) property:** $U(\exp(th)) = \Delta_{\mathsf{N}(W)}^{-it/2\pi}$ for $t \in \mathbb{R}$, $W \in \mathcal{W}_+$.

(HK6) **Haag Duality**: $\mathsf{N}(W') = \mathsf{N}(W)'$ if $W, W' \in \mathcal{W}_+$

If the representation U extends antiunitarily to G_{τ_h} we can further require:

(HK7) **G-covariance:** There exists an (anti-)unitary extension of U from G to G_{τ_h} and an extension of the net $\mathsf{N}(W)$ on $\mathcal{W} := \mathcal{W}(W_0)$, such that the following condition is satisfied:

$$\mathsf{N}(g.W) = U(g)\mathsf{N}(W) \quad \text{for} \quad g \in G_{\tau_h}, W \in \mathcal{W}$$

(HK8) **Modular reflection:** $U(\tau_W) = J_{\mathsf{N}(W)}$, for $W = (h_W, \tau_W) \in \mathcal{W}_+$

It is a consequence of (HK8) that Haag duality holds for $W, W' \in \mathcal{W}$.

We further indicate the modular covariance property as the geometric action of the modular group:

Wedge modular covariance: for every couple of wedges $W_a, W_b \in \mathcal{W}_+$,

$$\Delta_{\mathsf{N}(W_a)}^{-it} \mathsf{N}(W_b) = \mathsf{N}(\Lambda_{W_a}(2\pi t).W_b) \quad (18)$$

for $W_a, W_b \in \mathcal{W}_+$, $t \in \mathbb{R}$.

Note that it does not require the Bisognano–Wichmann property, but it holds whenever the latter is satisfied.

We remark that by identifying the abstract wedges with concrete wedge regions of Minkowski space, the net of standard subspaces on wedges of the free scalar fields given by (9) satisfies all the previous assumptions. The deep relation between wedge subspaces, wedge regions and the symmetry group representation has been made more explicit by the Brunetti–Guido–Longo construction as will be explained in the next section.

2.4.3 The Brunetti–Guido–Longo (BGL) net

The explicit construction of the free field we have seen in Sect. 2.4.1 for the free scalar field is highly non–canonical. It has been shown in [LMR16] that such a construction is not feasible for infinite-spin representations of the Poincaré group as they lack localized states in bounded regions. However, wedge subspaces are determined by the wedge symmetries. Specifically, given an (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+$, for any wedge $W \subset \mathbb{R}^{1,d}$, the wedge subspace is reconstructed as follows: The operators

$$\Delta_W^{it} := U(-2\pi h_W t), \qquad J_W := U(\tau_{h_W})$$

satisfy the commutation relation (15), hence $\Delta_W = \exp(2\pi i \partial U(h_W))$ and $J_W = U(\tau_{h_W})$ canonically define a the wedge standard subspace

$$\mathsf{H}(W) = \mathrm{Fix}(J_W \Delta_W^{\frac{1}{2}})$$

as we have seen in the previous section. Based on these remarks R. Brunetti, D. Guido and R. Longo in [BGL02] provided a construction of the one-particle net of standard subspaces of the free Poincaré and conformal covariant theories on Minkowski space, de Sitter space and the chiral circle on wedge regions. This construction has been generalized to our abstract set of wedges for a $\mathbb{Z}_2$-graded Lie group supporting Euler elements in [MN21]. The anti-de Sitter space can be included in the picture and it is treated in [NÓØ21, NÓ23], see also [Re00].

Let G be a connected Lie group with trivial center, such that $\mathcal{E}(\mathfrak{g}) \neq \{0\}$. We fix an Euler element and $h \in \mathfrak{g}$ and τ_h integrates to an involution on G. Let (U, G) be an (anti-)unitary representation of G_{τ_h}, $W = (h, \tau_h) \in \mathcal{G}_E$ an Euler wedge, and consider the couple of operators

$$J_{\mathsf{H}_U(W)} = U(\tau_h) \quad \text{and} \quad \Delta_{\mathsf{H}_U(W)} = e^{2\pi i \partial U(h)}. \tag{19}$$

They satisfy (15) and we can define $\mathsf{H}_U(W) := \mathrm{Fix}(J_{\mathsf{H}_U(W)} \Delta_{\mathsf{H}_U(W)}^{\frac{1}{2}})$. The map

$$\mathsf{H}_U^{\mathrm{BGL}} \colon \mathcal{G}_E(G_{\tau_h}) \ni W \to \mathsf{H}_U^{\mathrm{BGL}}(W) \in \mathrm{Stand}(\mathcal{H})$$

is the so-called BGL net associated to U and satisfies all the assumptions in Definition 2.1.

Theorem 2.2. *[MN21, Thm 4.12, Prop. 4.16] Let $(U, \mathcal{H})$ be an (anti-)unitary C-positive representation of G_τ where τ is an Euler involution. Then the BGL net*

$$\mathsf{H}_U^{\mathrm{BGL}} \colon \mathcal{G}(G_\tau) \to \mathrm{Stand}(\mathcal{H})$$

satisfies (HK1)-(HK8) in Definition 2.1.

The statement of this theorem can be more general as it is presented in [MN21]. Given a $\mathbb{Z}_2$-graded Lie group $G_\sigma = G \rtimes \{e, \sigma\}$ with σ an involution on G, then one can define more general abstract wedges as

$$\mathcal{G} := \{(h, \tau) \in \mathfrak{g} \times G\sigma : \mathrm{Ad}(\tau)(h) = h, \tau^2 = e\} \tag{20}$$

without referring to Euler elements, satisfying analogous properties to Euler wedges [MN21, Sect. 2] and the BGL net is still well defined. The reason why we will mainly deal with Euler element will become clear in Sect. 3.2.

We have introduced an axiomatic framework for nets of standard subspaces on wedges. This picture includes the well known models from AQFT and produces new examples. Indeed, all three graded Lie algebras support unitary representations of the $\mathrm{Inn}(\mathfrak{g})$-group that, given $h \in \mathcal{E}(\mathfrak{g})$, extend to G_{τ_h}. Up to coupling conjugate representations, U extends to an (anti-)unitary representation of G_{τ_h} and one can construct the BGL net of standard subspaces on abstract wedge regions. Hermitian simple Lie algebras have a proper positive cone and their unitary positive energy representations are studied in [MdR07]. Once a net of standard subspaces on abstract wedge regions is constructed the second quantization net of von Neumann algebras is canonically provided by relations (10) and (11). Furthermore, Theorem 2.2 also works if $C = C_U = \{0\}$. Then positivity of the energy is trivially satisfied as well as the isotony property. We stress that if the group is not centerfree the locality property has to be rewritten using twisted central complements. This has been investigated in [MN21, Sect. 4.2.2]. In this case a proper second quantization procedure respecting a twisted locality conditions has still to be established. Recent results on second quantizations are contained in [CSL23].

We have seen that there is a correspondence between abstract wedges and concrete wedges on causal manifolds. It is possible to provide a construction for a net of standard subspaces directly on the causal manifold extending the construction presented for the free field on Minkowski space.

Given a unitary representation $(U, \mathcal{H})$ of a connected a Lie group G and a homogeneous space $M = G/H$, we are interested in families $(\mathsf{H}(\mathcal{O}))_{\mathcal{O} \subseteq M}$ of closed real subspaces of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$, the so-called *nets of real subspaces on M*.

This setting is depicted by the following set of axioms:

(Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$

(Cov) **Covariance:** $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.

(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.

(BW) **Bisognano–Wichmann property:** There exists an open subset $W \subseteq M$ (called a *wedge region*), such that $\mathsf{H}(W)$ is standard with modular operator $\Delta_{\mathsf{H}(W)} = e^{2\pi i \partial U(h)}$ for some $h \in \mathfrak{g}$.

Nets satisfying (Iso), (Cov), (RS), (BW) on non-compactly causal symmetric spaces have been constructed in [FNÓ23], and on compactly causal spaces in [NÓ23] using the language of distribution vectors. Given the G-orbit of the wedge defined, one can define the *maximal net* on bounded regions by wedge intersections $\mathsf{H}^{\max}(\mathcal{O}) = \bigcap\{\mathsf{H}(gW) : \mathcal{O} \subset gW, g \in G\}$, this has been studied in [MN24, Sect. 2.2.4]. We remark that locality is not present among the list, since it is to be investigated [MN25].

3 Euler elements, Bisognano–Wichmann property and algebra types

3.1 Non-modular covariant nets of standard subspaces

A long standing question in AQFT concerns the necessity to assume the Bisognano–Wichmann property. In particular, given an isotonous, covariant, local, (positive energy) net of von Neumann algebras, does the Bisognano–Wichmann property hold? This is always the case if the local algebra net is generated by Wightman fields as it is proved in [BW75]. It holds in conformal theories ([Lo08, BGL93]) and in massive theories [Mu01]. In some scalar case one can give an algebraic condition ensuring the Bisognano–Wichmann property that applies also in possibly interacting theories [Mo18, DM20]. When infinite particle degeneracy is present many counterexamples to Bisognano-Wichmann property arise [Mo18, MT19]. On the other hand in these counterexamples the wedge modular group implements a covariant representation of the symmetry group. In [Yn94], Yngvason provided a translation covariant von Neumann algebra net where there is no geometric modular action of the wedge algebra. On the other hand these nets are not expected to be Lorentz covariant.

Since the modular theory of a net of von Neumann algebras in the vacuum representation is contained in the standard subspace structure of wedge algebras, it is a natural problem to provide counterexamples to the Bisognano–Wichmann property in the net of standard subspace context without modular covariance on wedge subspaces. This problem has been recently faced in [MN22] providing a family of non-local standard subspace nets with no modular action of the wedge modular group. Here we point out the main steps of the construction.

Let $G = \mathrm{Inn}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$, let G be the graded extension $G_\tau = G \rtimes \{1, \tau\}$, where τ is an Euler involution and let U be an (anti-)unitary representation of G_τ on a Hilbert space $\mathcal{H}$. Assume the existence of a subgroup $H_\tau = H \rtimes \{e, \tau\} \subseteq G_\tau$, two Euler wedges $W_1 = (h_1, \tau_1) \in \mathcal{G}_E(H_\tau)$ and $W_2 = (h_2, \tau_2) \in \mathcal{G}_E(G_\tau)$. Note that h_1 is not necessarily in $\mathcal{E}(\mathfrak{g})$. Assume that the stabilizer H_{W_1} of W_1 in H fixes W_2, in particular $[h_1, h_2] = 0$. We denote the orbit $H.W_1$ with $\mathcal{W}_+(H, W_1)$.

Consider the BGL construction with respect to U, G_τ and $\mathcal{G}_E(G_\tau)$. The standard subspace $\mathsf{N}_2 = \mathsf{N}_U(W_2)$ is defined by

$$J_{\mathsf{N}_2} = U(\tau_2) \quad \text{and} \quad \Delta_{\mathsf{N}_2} = e^{2\pi i \cdot \partial U(h_2)}.$$

Since the stabilizer H_{W_1} fixes W_2 and the unitary group $U(H_{W_1})$ fixes N_2, and we can define an H-equivariant map

$$\mathsf{N}\colon \mathcal{G}_E(H_\tau) \supseteq \mathcal{W}_+ := \mathcal{W}_+(H, W_1) := H.W_1 \to \mathrm{Stand}(\mathcal{H}), \quad g.W_1 \mapsto U(g)\mathsf{N}_2$$

which is uniquely determined by

$$\mathsf{N}(W_1) = \mathsf{N}_2. \tag{21}$$

The net N is not necessarily modular covariant as the modular groups of the wedge subspaces $U(g)\mathsf{N}_2$ are in general subgroups of the larger group G. The following is an equivalent condition for the modular covariance property (18) (cf. [MN22, Lem 3.1]). The net N on $\mathcal{W}_+(H, W_1)$ satisfies modular covariance if and only if, for all $g \in H, t \in \mathbb{R}$, the operator

$$U(g)U(\exp t(h_1 - h_2))U(g)^{-1}$$

fixes the standard subspace N_2, i.e.,

$$g \exp(t(h_1 - h_2))g^{-1} \in G_{\mathsf{N}_2} \quad \text{for} \quad g \in H, t \in \mathbb{R}. \tag{22}$$

If $\ker(U)$ is discrete, then (22) is violated if

$$[\mathfrak{h}, h_1 - h_2] \not\subseteq \ker(\mathrm{ad}\ h_2), \tag{23}$$

where $\mathfrak{h} = \mathbf{L}(H)$, see [MN22, Rem. 3.2].

We now indicate how to use this prescription to construct a non-modular covariant net on the two-dimensional de Sitter space. Here the key choice is to take $h \in \mathfrak{g}$ to be a non-symmetric Euler element in $\mathfrak{g}$ and check that (22) is not satisfied by testing (23). We refer to [MN22] for the general discussion and further examples as non-modular covariant nets on Minkowksi space and the representation theoretic results.

Consider the case $G = \mathrm{Inn}(\mathfrak{g})$ where $\mathfrak{g}$ a simple non-compact Lie algebra. Consider a **non-symmetric** Euler element $h_2 \in \mathfrak{g}$ and the associated wedge $W_2 = (h_2, \tau_2) \in \mathcal{G}_E(G_\tau)$. By [MN22, Lem. 2.15] there exists a $\mathfrak{gl}_2$-subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing h_2 such that $\mathfrak{h} := [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$ satisfies $[\mathfrak{h}, h_2] \neq 0$. We consider $H := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \cong \mathrm{SL}_2(\mathbb{R})$ and $H_{\tau_2} = H \rtimes \{1, \tau_2\}$ (see [MN22, Lem 2.15(d)]). We assume the Euler element $h_2 \in \mathfrak{b}$ to have a central component h_c, so that

$$h_2 = h_c - h_1 \quad \text{with} \quad h_c \in \mathfrak{z}(\mathfrak{b}) \quad \text{and} \quad h_1 \in \mathcal{E}(\mathfrak{h}). \tag{24}$$

Choosing the isomorphims $\mathfrak{h} \to \mathfrak{sl}_2(\mathbb{R})$ suitably, we find

$$h_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

Let V be the restriction $U|_{H_{\tau_2}}$. The group $H \cong \mathrm{SL}_2(\mathbb{R})$ is the double covering of $\mathrm{PSL}_2(\mathbb{R}) \cong \mathcal{L}_+^\uparrow$, the Lorentz group of $\mathbb{R}^{1,2}$. So it acts on de Sitter space dS^2 through the covering map $\Lambda : \mathrm{SL}_2(\mathbb{R}) \ni g \mapsto \Lambda(g) \in \mathrm{PSL}_2(\mathbb{R})$. We thus obtain a net H^{dS} on de Sitter spacetime as follows [MN22, Thm. 3.4]: Let

$$\mathcal{W}^{\mathrm{dS}} \ni W^{\mathrm{dS}} \mapsto \mathsf{H}^{\mathrm{dS}}(W^{\mathrm{dS}}) \subset \mathcal{H}_U$$

such that

$$\mathsf{H}^{\mathrm{dS}}(\Lambda(g)W_R^{\mathrm{dS}}) := V(g)\mathsf{N}_U(W_2) \quad \text{for} \quad g \in H \cong \mathrm{SL}_2(\mathbb{R}), \tag{26}$$

where N_U is the BGL net defined by U. Then the net H^{dS} is Lorentz covariant and does not satisfy modular covariance. Indeed $h_2 - h_1 = h_c - 2h_1$ and

$$[h_2 - h_1, \mathfrak{h}] = [h_1, \mathfrak{h}] \not\subseteq \ker(\mathrm{ad}\ h_1).$$

In particular (23) is violated and N is not modular covariant. One can also see this by observing that the Lie algebra generated by gh_2g^{-1} with $g \in \mathrm{SL}_2(\mathbb{R})$ is $\mathfrak{gl}_2(\mathbb{R})$ so that $\Delta^{it}_{\mathsf{H}^{\mathrm{dS}}(gW_R^{\mathrm{dS}})} = U(\exp(2\pi g h_2 g^{-1} t))$ can not generate a representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Details in [MN22, Sect. 3.2.1 and 3.2.2]. We can further comment that counterexamples satisfying Lorentz covariance, locality and violating modular covariance are not covered by these examples and are still missing.

3.2 The Euler Element Theorem

The following theorem is an important step in understanding the key role of Euler elements in these constructions and their relation with localization properties of a net of standard subspaces or von Neumann algebras. It addresses the question, what does characterize the boosts to be such a special family of transformations that allows the Bisognano–Wichmann property to start all this discussion? Given a unitary representation of a Lie group and a standard subspace V satisfying the

Bisognano–Wichmann property with respect to a generic one-parameter group in G, the generator of the latter in U is an Euler element if a regularity property is satisfied. The following Theorem ensures that choosing Euler element is the only choice under natural conditions (cf. equation (20)).

Theorem 3.1. (Euler Element Theorem) *[MN24, Thm. 3.1] Let G be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of G with discrete kernel. Suppose that $\mathtt{V}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that the following hold:*

(a) ***Bisognano–Wichmann property:*** $U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ *for* $t \in \mathbb{R}$, *i.e.,* $\Delta_{\mathtt{V}} = e^{2\pi i\, \partial U(h)}$.

(b) ***Regularity property:*** $\mathtt{V}_N := \bigcap_{g\in N} U(g)\mathtt{V}$ *is cyclic.*

Then h is an Euler element and the conjugation $J_{\mathtt{V}}$ satisfies

$$J_{\mathtt{V}} U(\exp x) J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{27}$$

The theorem implies that, under the above assumptions, if $\mathfrak{g}$ is a compact Lie algebra, then every Euler element $h \in \mathfrak{g}$ is central, so that $\tau_h = \mathrm{id}_{\mathfrak{g}}$. Therefore, a standard subspace $\mathtt{V}$ associated to a pair $(h, \tau) \in \mathcal{G}_E$ by the BGL construction can only satisfy the regularity condition in Theorem 3.1(b) if $\mathtt{V}$ is $U(G)$-invariant.

It is a consequence of Theorem 3.1 that τ_h integrates to an involutive automorphism τ_h on the group $U(G) \cong G/\ker(U)$ that is uniquely determined by

$$\tau_h(\exp x) = \exp(\tau_h(x)) \quad \text{for} \quad x \in \mathfrak{g}.$$

In particular the theorem ensures the extendibility of U to an (anti-)unitary representation of G_{τ_h} on the same Hilbert space if U is a faithful representation of G satisfying Bisognano–Wichmann and regularity properties.

The regularity property is very natural in the setting of an analysis of state localization in AQFT, see for instance [BGL02, GL95, LMR16]. Firstly note that the condition (b) is equivalent to the existence of a cyclic subspace $\mathsf{K} \subset \mathsf{H}$ such that $U(g)\mathsf{K} \subset \mathtt{V}$ for every $g \in N$. This corresponds to a localization property of standard subspace nets that are finer than wedges. To provide examples we firstly refer to theories on Minkowski spacetime. We call an open subset $\mathcal{O} \subset \mathbb{R}^{1,d}$ *spacelike* if $x_0^2 < \mathbf{x}^2$ holds for all $(x_0, \mathbf{x}) \in \mathcal{O}$ and a spacelike open subset is called a *spacelike (convex) cone* if, in addition, it is a pointed convex cone. Given a spacelike cone $\mathcal{C}$ with apex in $a \in \mathbb{R}^{1,d}$ then $\mathcal{C} = \bigcap\{gW_R + a : g \in \mathcal{L}_+^\uparrow, gW_R \supset \mathcal{C} - a\}$ and for every wedge W there always exists a spacelike cone $\mathcal{C} \subset gW$, for every g in some small enough neighbourhood of the identity $N \subset \mathcal{P}_+^\uparrow$. Let H be a net of real subspaces on open regions satisfying isotony, covariance, cyclicity on wedge regions, then the assumption (a) and (b) in the theorem correspond respectively to the Bisognano–Wichmann property and the cyclicity for $\mathsf{H}(\mathcal{C})$. This is always the case for the free one-particle nets on Minkowksi space when $\mathcal{C}$ is a spacelike cone, see [BGL02, Sect. 4]. One can make an analogue remark for de Sitter spacetime. One can consider the intersection of pointed convex spacelike cones with apex in the origin in $\mathbb{R}^{1,d}$ with de Sitter space dS^{d} and $\mathcal{C} \cap \mathrm{dS}^{\mathrm{d}}$ are bounded regions obtained by wedge intersection.

Another remark, in view of [DM20] there exists no net of standard subspaces on spacelike cone regions undergoing an irreducible massless finite (non-zero) helicity $\mathcal{P}_+^\uparrow$-representation U. The argument can be seen as a consequence of Theorem 3.1 since U should then extend to a representation of $\mathcal{P}_+$ on the same Hilbert space. This is not possibile for finite helicity representation, cf. [Va85, Thm. 9.10].

The Euler Element Theorem can be applied in the setting of nets of standard subspaces on causal manifolds, see [MN24, Thm. 3.4]. With notations as in Sect. 2.4.3, let $(U, \mathcal{H})$ be a unitary representation of the connected Lie group G with $\ker(U)$ discrete. If $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ is a net of real subspaces on (the open subsets of) a G-manifold M that satisfies (Iso), (Cov), (RS) and (BW), then Theorem 3.1 applies to $\mathsf{H}(W)$, and the Lie algebra element h is an Euler element, and the conjugation $J := J_{\mathsf{H}(W)}$ satisfies (27).

The theorem applies also when von Neumann algebras are taken into account. Let G, h and $(U, \mathcal{H})$ be as in Theorem 3.1. Let Ω be a unit vector and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra for which Ω is cyclic and separating. Assume that Ω is fixed by $U(G)$ and that the Bisognano–Wichmann property holds as $U(\exp(-2\pi th)) = \Delta^{it}_{\mathcal{M},\Omega}$. Assume that for some e-neighborhood $N \subseteq G$, the vector Ω is still cyclic for the von Neumann algebra

$$\mathcal{M}_N := \bigcap_{g\in N} \mathcal{M}_g, \qquad \text{where} \qquad \mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1}.$$

Then h is an Euler element and the modular conjugation $J = J_{\mathcal{M},\Omega}$ of the pair $(\mathcal{M}, \Omega)$ satisfies

$$JU(\exp x)J = U(\exp \tau_h(x)) \quad \text{ for } \quad \tau_h = e^{\pi i \operatorname{ad} h}.$$

This is a conclusion obtained by applying Theorem 3.1 to the standard subspace $\mathsf{H} = \overline{\mathcal{M}_{sa}\Omega}$, see [MN24, Thm. 3.7].

Given a local, Lorentz covariant, weakly additive net of von Neumann algebras on bounded regions of de Sitter space the authors of [BB99, Theorem 6.2], starting from the assumption that vacuum states in de Sitter space, look for any geodesic observer like equilibrium states with some a priori arbitrary temperature, show that the geodesic temperature of vacuum states has to be equal to the Gibbons–Hawking temperature. This corresponds to the fact that, under the Bisognano–Wichmann and regularity properties, the boost generator is an Euler element and its associated one-parameter group - properly parameterized - implements the modular group through the representation U. In the positive energy framework an analogous statement has been proved for the vacuum state in [St08, Thm. 1]. The Euler Element Theorem gives a standard subspace reformulation of these results [MN24, Cor. 3.3 and 3.5].

3.3 Regularity and Localizability

Given a connected Lie group G, such that $\mathcal{E}(\mathfrak{g}) \neq \{0\}$, a $\mathbb{Z}_2$-graded extension G_{τ_h} with respect to an Euler involution τ_h, where $h \in \mathcal{E}(\mathfrak{g})$, and an (anti-)unitary representation U of G_{τ_h}, one can construct a net of standard subspaces on wedges as in Sect. 2.4.3. The Bisognano–Wichmann property holds, which naturally raises the question of the extent to which the regularity property (b) from Theorem 3.1 is also satisfied. This is a representation theoretic question investigated in [MN24] that in part uses the results of [FNÓ23].

In this setting we reformulate the property as follows. Assuming that h is an Euler element and $(U, \mathcal{H})$ an (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h}. Let $\mathtt{V}$ be the standard subspace associated to the wedge (h, τ_h) by the BGL–construction. We shall say that U is *regular with respect to* h, or *h-regular*, if there exists an e-neighborhood $N \subseteq G$ such that $\mathtt{V}_N = \bigcap_{g\in N} U(g)\mathtt{V}$ is cyclic. Replacing N by its interior, we may always assume that N is open.

This property passes to direct sums, direct integrals and sub-representations as follows [MN24, Lem. 4.4]:

(a) If $U = U_1 \oplus U_2$ is a direct sum, then U is h-regular if and only if U_1 and U_2 are h-regular.

(b) If U is h-regular, then every subrepresentation is h-regular.

(c) Assume that G has at most countably many components. Then a direct integral $U = \int_X^\oplus U_m \, d\mu(m)$ is regular if and only if there exists an e-neighborhood $N \subseteq G$ such that, for μ-almost every $m \in X$, the subspace $\mathtt{V}_{m,N}$ is cyclic.

We now present the state of the art on the analysis of this property:

Regularity via positive energy

In conformal Algebraic Quantum Field Theory, the geometry of the model ensures that one can shrink wedge regions: conformal symmetries in Minkowski space can transform a wedge region W into a doublecone region $O \Subset W$ and both regions are associated 1-1 to Euler elements; for Möbius symmetries of the circle, intervals are associated to Euler wedges, and interval inclusions are associated to inclusions of Euler wedges [BGL93, BGL02, MN21]. In particular, provided the BGL construction for these models, there exists an open neighborhood of the identity $N \subset G$ such that given a $\bigcap_{g\in N} gW_1$ contains a wedge region supporting a cyclic subspace, hence regularity for the BGL net holds.

Firstly, we situate the general framework within the Lie theory context, where the regularity property stems from the existence of a generating positive cone and the positivity of the energy. Theorem [MN24, Thm. 4.9] claims that, if $(U, \mathcal{H})$ is an (anti-)unitary representation of G_{τ_h} for which the cones

$$C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

are linearly generating $\mathfrak{g}_{\pm 1}(h)$, then $(U, \mathcal{H})$ is regular.

As examples one can consider the Möbius group. Here, the condition on the cone $C_\pm$ to be generating holds for positive energy representations of the Möbius group. Up to sign, the only pointed, generating closed convex Ad-invariant cone has been introduced in (5). Given a positive energy representation U of $\mathsf{M\ddot{o}b}$, then $C_U = C$. Considering the Euler element $h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have

$$C_\pm = \pm C \cap \mathfrak{g}_{\pm 1}(h), \quad C_+ = \mathbb{R}_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_- = \mathbb{R}_+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the half lines $C_\pm$ generate $\mathfrak{g}_{\pm 1}(h)$. This property is not satisfied by positive energy representations of the Poincaré group on $\mathbb{R}^{1,3}$. We will see in the next section how to deal with this case.

Another example is the the following. Let $\mathfrak{g} = \mathfrak{so}(2, d+1)$ the Lie algebra of the conformal group of the Minkowski spacetime $\mathbb{R}^{1,d}$, consider the Euler element $h \in \mathfrak{g}$ generating the dilations $\exp(ht)x = e^t x$ where $x \in \mathbb{R}^{1,d}$, then $\mathfrak{g}_1 \simeq \mathbb{R}^{1,d}$ and the positive cone C is generating in $\mathfrak{g}_1$ (and analogously for $\mathfrak{g}_{-1}$). The generating property holds for all hermitian simple real Lie algebras, see [NÓØ21, Lem. 3.2].

Regularity in a semidirect product

As a second step, it looks natural to study the regularity property for (anti-)unitary representation of semidirect products. This picture now contains the Poincaré group that is not simple. In general, a simply connected Lie group G, is a semidirect product $G \cong R \rtimes S$, where S is semisimple and R is the solvable radical (Levi decomposition, see e.g. [Ho81, Thm. VIII.4.3]).

One can split the problem checking two conditions on the two subgroups defining the semidirect product. Consider a semidirect product Lie group $G = R \rtimes L$, let $0 \neq h \in \mathcal{E}(\mathfrak{g})$, and $(U, \mathcal{H})$ be an (anti-)unitary representation of G_{τ_h}. Theorem 4.11 in [MN24] proves that $(U, \mathcal{H})$ is regular under the following two conditions

(a) the cones $C_\pm := \pm C_U \cap \mathfrak{r}_{\pm 1}(h)$ generate $\mathfrak{r}_{\pm 1}(h)$ where $\mathfrak{r} = \mathbf{L}(R)$.

(b) $(U|_L, \mathcal{H})$ is regular.

This theorem applies to (anti-)unitary positive energy representation of the Poincaré group $\mathcal{P}_+$ as follows: look at the representation of the Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+^\uparrow$, check the non-triviality of the one-dimensional cones $C_\pm$ in the eigenspaces $\mathfrak{r}_{\pm 1}(h_{W_R})$ that corresponds to the lightrays $\mathbb{R}(\mathbf{e}_0 \pm \mathbf{e}_1)\mathfrak{r} \cong \mathbb{R}^{1,d}$ and the regularity property for the restriction of the representation to the identity component $\mathcal{L}_+^\uparrow$ of the Lorentz group. The first property corresponds to the positive energy condition on the Poincaré representation. The second one holds for every representation of the Lorentz group, as one can see from Theorem 4.25 in [MN24]. Note that one can relate this special case of the results in the Appendix of [BGL02, Thm 4.7]

To study the regularity property for reductive Lie groups G one can introduce the localizability property. Given an (anti-)unitary representation $(U, \mathcal{H})$ of the Lie group G, this property ensures the existence of a concrete cyclic subspace of the representation Hilbert space described in terms of distribution vectors of the Hilbert space $\mathcal{H}$, localized on an open region of a causal symmetric manifold M lying in the intersection of wedge subspaces (wedge regions as described in Sect. 2.3). One explicitly constructs local subspaces (in analogy with the the free field case) and proves the cyclicity of subspaces associated to intersections of wedge regions and the Bisognano–Wichmann property for standard subspace of wedge regions of the causal manifold. This has been studied in [FNÓ23] and recalled in this setting to conclude regularity in [MN24]. Results in [MN24, Thm. 4.23, Cor 4.24] guarantee localizability for representations of a connected reductive Lie group. Then the above theorem applies to the cases where $G = R \rtimes L$, $U|_R$ satisfies (a) and L is reductive.

Currently, it is unknown whether all (anti-)unitary representations of Lie groups of the form G_{τ_h}, where $h \in \mathcal{E}(\mathfrak{g})$ is an Euler element, are regular. The preceding discussion suggests that resolving this question requires a more detailed analysis of the case of solvable groups.

3.4 The type of wedge algebras

Von Neumann algebras can be classified in terms of the geometry of their projection and the spectrum of the modular action [Ta02, Su87, Co73]. Algebras taking part in an AQFT are proved to be type III_1 in many cases and it is expected to be always the case when they are nontrivial (see for instance [Dr77, Lo82, Fr85, BDF87, BB99, FG94]). Recently, type II_1 algebras have become increasingly relevant in certain constructions within the AQFT context, see [CLPW23, FJLRW25].

We prove that our generalized AQFT setting also respects this expectation. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We say that h is anti-elliptic if $\mathfrak{n}_h + \mathbb{R}h = \mathfrak{g}$ where $\mathfrak{n}_h \trianglelefteq \mathfrak{g}$ be the smallest ideal of $\mathfrak{g}$ such that the image of h in the quotient Lie algebra $\mathfrak{g}/\mathfrak{n}_h$ is elliptic, namely $\mathrm{ad}\, x$ is semisimple with purely imaginary spectrum. Let $(U, \mathcal{H})$ be a unitary representation of G with discrete kernel, $\mathcal{N} \subset \mathcal{M} \subseteq B(\mathcal{H})$ an inclusion of von Neumann algebras, and $\Omega \in \mathcal{H}$ a unit vector which is cyclic and separating for $\mathcal{N}$ and $\mathcal{M}$. Assume that the following properties hold:

(Mod) $e^{2\pi i \partial U(h)} = \Delta_{\mathcal{M},\Omega}$, and

(Reg) $\{g \in G\colon \mathrm{Ad}(U(g))\mathcal{N} \subseteq \mathcal{M}\}$ is an e-neighborhood in G.

Then the following assertions hold:

(a) h is an Euler element.

(b) The conjugation $J := J_{\mathcal{M},\Omega}$ satisfies

$$JU(\exp x)J = U(\exp \tau_h(x)) \quad \text{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{28}$$

(c) $\mathcal{H}^G = \ker(\partial U(h))$.

(d) The restriction of $i\partial U(h)$ to the orthogonal complement of the subspace $\mathcal{H}^{N_h}$ of fixed vectors of the codimension-one normal subgroup $N_h := \langle \exp(\mathfrak{n}_h) \rangle \trianglelefteq G$, has absolutely continuous spectrum.

If, in addition, $\mathcal{H}^G = \mathbb{C}\Omega \neq \mathcal{H}$, then $\mathcal{M}$ is factor of type III_1. This theorem is [MN24, Thm. 5.15].

Note that if $h \in \mathcal{E}(\mathfrak{g})$ is anti-elliptic, then $\mathfrak{g}_0 \subseteq \mathbb{R}h + [\mathfrak{g}_1, \mathfrak{g}_{-1}]$. Anti-elliptic elements are referred to as essential element in [St08]. One can apply this setting when it is considered a unitary representation U on an Hilbert space $\mathcal{H}$ of the connected Lie group G with discrete kernel, a vector $\Omega \in \mathcal{H}$ that is the unique unit vector fixed by $U(G)$ and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra for which Ω is cyclic and generating. Assume that $h \in \mathfrak{g}$ is anti-elliptic and the following properties.

(Mod) **Modularity:** There exists an element $h \in \mathfrak{g}$ for which $e^{2\pi i \partial U(h)} = \Delta_{\mathcal{M},\Omega}$. As $\ker(U)$ is discrete, h is uniquely determined.

(Reg) **Regularity:** For some e-neighborhood $N \subseteq G$, the vector Ω is still cyclic (and obviously separating) for the von Neumann algebra

$$\mathcal{M}_N := \bigcap_{g \in N} \mathcal{M}_g, \quad \text{where} \quad \mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1}.$$

Then h is an Euler element and $\mathcal{M}$ is a type III_1 factor. This picture includes the cases of nets of von Neumann algebras on Euler wedges when, under the previous assumptions,

$$\mathcal{W}_+ \ni W \mapsto \mathcal{M}(gW) = U(g)\mathcal{M}(W_h)U(g)^{-1} \subset \mathcal{B}(\mathcal{H}), \qquad \mathcal{M}(W_h) = \mathcal{M},\, W_h = (h, \tau_h),\, g \in G$$

defines a G-covariant isotonous net of von Neumann algebras.

In the case Ω is cyclic and separating but the set of $U(G)$-fixed points is not one dimensional, then one can consider the von Neumann algebra $\mathcal{A} := \big(\bigcup_{g \in G} \mathcal{M}_g\big)'' \subseteq B(\mathcal{H})$ generated by all algebras $\mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1}$. If $\mathcal{M}' = \mathcal{M}_{g_0}$ for some $g_0 \in G$ then we have direct integral decompositions

$$\mathcal{M} = \int_X^{\oplus} \mathcal{M}_x \, d\mu(x), \qquad U = \int_X^{\oplus} U_x \, d\mu(x), \qquad \text{and} \qquad \mathcal{A} = \int_X^{\oplus} B(\mathcal{H}_x) d\mu(x).$$

We have a measurable decomposition $X = X_0 \dot\cup X_1$, where $\dim \mathcal{H}_x = 1$ for $x \in X_0$ and the representations $(U_x)_{x \in X_0}$ are trivial. For $x \in X_1$, the algebras $\mathcal{M}_x$ are factors of type III_1 and $(\mathcal{M}_x, \Omega_x, \underline{U}_x)$ satisfies (Reg) and (Mod), where $\underline{U}_x$ is the representation of $G/\ker(U_x)$ induced by U_x, further details in [MN24, Thm. 5.22].

4 An outlook on hermitian Lie algebra

We have seen that various aspects of the geometric framework of Algebraic Quantum Field Theory (AQFT) can be explored at the Lie theory level. In particular, the Bisognano–Wichmann (BW) property and the modular reflection lead to a wedge-boost generalizing correspondence when a Lie group symmetry provided with an Euler element of the Lie algebra is considered. As we have seen along these lines, this framework allows for the deduction of properties of wedge regions, wedge symmetries, wedge standard subspaces and wedge von Neumann algebras.

Within this generalized setting, conformal models stand out due to the significant role played by orthogonal wedges, offering a richer structure [MN21]. Here the Lie algebra of the symmetry group is hermitian, namely a Lie algebra such that the center of a maximal compactly embedded subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is non-zero. A simple real Lie algebra $\mathfrak{g}$ is hermitian if and only if it has a convex $\mathrm{Inn}(\mathfrak{g})$-invariant cone $C \neq \{0\}$ or $\mathfrak{g}$. Furthermore a simple real Lie algebra containing Euler elements is hermitian if and only if it is of tube type, i.e. the restricted root system is of type C_r (see [MN21, Prop.3.11]). In view of the classification in Sect.2.2, in this case, there exists a unique orbit of symmetric Euler elements. In [MN21, Theorem 3.13] in a simple Lie algebra symmetric Euler elements have orthogonal Euler element partners. On Minkowksi space this is depicted by wedges in orthogonal directions, as $W_i = \{x \in \mathbb{R}^{1,d}, |x_0| < x_i\}$ with $i = 1, 2$, where $j_{W_1}\Lambda_{W_2}(t)j_{W_1} = \Lambda_{W_2}^{-1}(t)$. In our language orthogonal Euler elements are the Lie elements defining orthogonal wedges, satisfy $\tau_{h_1}(h_2) = -h_2$, where $h_1, h_2 \in \mathcal{E}(\mathfrak{g})$.

One can consider couples of orthogonal elements and study orbits of orthogonal elements. The existence of disjoint orbits is particularly relevant in Conformal Field Theory on Minkowksi spacetime. Given a unitary representation U of the universal covering of the conformal group $\widetilde{G}$, where $\mathfrak{g} = \mathfrak{so}(2, d+1)$, inequivalent couples of orthogonal Euler elements generate inequivalent representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ not all of them for instance have positivity of the energy. A systematic analysis of these orthogonal pairs, along with a geometric examination of the conformal models, will be discussed in forthcoming papers together with an analysis of their relationship with conformal AQFT models ([MNO25, MN25]).

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