Deformations of Operator Algebras and the Construction of Quantum Field Theories

Gandalf Lechner

Department of Physics, University of Vienna

AQFT – The first 50 years, Uni Göttingen

Algebraic Quantum Field Theory

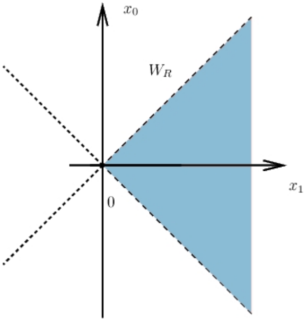
- Algebraic QFT has in the past mainly focussed on the analysis of general, model-independent properties of quantum field theories / nets of algebras
- Many tools to extract physical data from a given net are available today (particle content, cross sections, charges, short distance behaviour, and many more ...)
- But, as in any approach to QFT, the rigorous construction of models is still a challenging problem (in particular in $d = 4$)

Some Constructive Approaches to QFT

- In perturbative setting, use classical Lagrangean as input, then perturbative renormalization [→ Fredenhagen's talk]
- Good quantum description of possible interactions still missing
- Exception: Integrable models in $d = 2$ with S-matrix simple enough to be taken as an input [Schroer 97-01, GL 03, Buchholz/GL 04, GL 08]
- In algebraic QFT, individual models can be desribed by algebraic data (i.e. half-sided inclusions for conformal QFTs on the circle) [→ Longo's talk]
- In this talk, focus on the construction of models on $\mathbb{R}^d$ without conformal symmetry.

Wedges

In the following, wedge regions play a significant role.



- The right wedge

$$W_R := \{x \in \mathbb{R}^d \,:\, x_1 > |x_0|\}$$

  General wedge: Poincaré transform $W = \Lambda W_R + x$.

- "Wedges are big enough to allow for simple observables being localized in them, but also small enough so that two of them can be spacelike separated"

Local nets and wedge algebras

Local nets can be constructed from a single algebra ("wedge algebra") and an action of the Poincaré group.

- Let $\mathcal{B}$ a C^*-algebra with automorphic Poincaré action α and C^*-subalgebra $\mathcal{A} \subset \mathcal{B}$ such that

$$\alpha_{x,\Lambda}(\mathcal{A}) \subset \mathcal{A} \qquad \text{for } (x,\Lambda) \text{ with } \Lambda W_R + x \subset W_R \qquad \text{"isotony condition"}$$
$$\alpha_{x,\Lambda}(\mathcal{A}) \subset \mathcal{A}' \qquad \text{for } (x,\Lambda) \text{ with } \Lambda W_R + x \subset W_R' \qquad \text{"locality condition"}$$

The system $\mathcal{A} \subset \mathcal{B}$, α will be called a wedge algebra.

- Then

$$\Lambda W_R + x \longmapsto \alpha_{x,\Lambda}(\mathcal{A})$$

is a well-defined, isotonous, local, covariant net of C^*-algebras.

- Extension to smaller regions: $\mathcal{A}(\bigcap W_n) := \bigcap \mathcal{A}(W_n)$.

Task

Given a wedge algebra $\mathcal{A} \subset \mathcal{B}$, α, (e.g. given by an interaction-free theory) satisfying the isotony and locality condition, construct a new wedge algebra $\hat{\mathcal{A}} \subset \hat{\mathcal{B}}, \hat{\alpha}$ still satisfying these conditions, such that the associated net has non-trivial S-matrix.

- Deform $\mathcal{A} \subset \mathcal{B}$, α continuously from the free to the interacting case ("Perturbation theory for wedge algebras")
- Keep α fixed (scattering theory)

Wedge-local Deformations in QFT

Development of the subject:

- Deformation of free field theories on Minkowski space by transferring them to "noncommutative Minkowski space" (CCR techniques) [Grosse/GL 07]
- Generalization of this procedure to arbitrary QFTs by "warped convolutions" in an operator-algebraic setting [Buchholz/Summers 08]
- Deformation of Wightman QFTs by introducing a new product on the Borchers-Uhlmann testfunction algebra [Grosse/GL 08] [$\rightarrow$ Yngvason's talk]
- Connection between these two points of view: New product in the operator-algebraic setting $\rightarrow$ Rieffel deformations [Buchholz/Summers/GL, work in progress]

Rieffel Deformations

- Deformation procedure for C^*-algebras [Rieffel 93]
- Inspired by quantization, "strict deformation quantization"
- Setting: C^*-algebra $\mathcal{B}$ with strongly continuous automorphic action β of $\mathbb{R}^d$.
- Deformation parameter: antisymmetric real $(d \times d)$-matrix θ
- On dense subalgebra $\mathcal{B}^\infty \subset \mathcal{B}$ of smooth elements, define new product

$$A \times_\theta B := (2\pi)^{-d} \int dp \int dx \, e^{-ipx} \, \beta_{\theta p}(A)\beta_x(B)$$

  Integral defined in an oscillatory sense
- This product was designed to deform a commutative C^*-algebra $\mathcal{B}$ into a noncommutative one, but it can also be applied to noncommutative $\mathcal{B}$.

Rieffel Deformations

$$A \times_\theta B := (2\pi)^{-d} \int dp \int dx\, e^{-ipx}\, \beta_{\theta p}(A)\beta_x(B)$$

Main results about the product $\times_\theta$ [Rieffel 93]:

- $A \times_0 B = AB$
- $\times_\theta$ is an associative product on $\mathcal{B}^\infty$
- $(A \times_\theta B)^* = B^* \times_\theta A^*$
- $A \times_\theta 1 = A = 1 \times_\theta A$
- β is still automorphic w.r.t. $\times_\theta$.
- smooth algebra $\mathcal{B}_\theta^\infty$ can be completed to a deformed C^*-algebra $\mathcal{B}_\theta$

States and representations

- Deformation $\mathcal{B} \to \mathcal{B}_\theta$ introduces new positive cone,

$$B^*B \in \mathcal{B}^+ , \qquad (B^* \times_\theta B) \in \mathcal{B}_\theta^+ .$$

- A state on $\mathcal{B}$ is usually only a linear functional on $\mathcal{B}_\theta$
- Each state on $\mathcal{B}$ can be deformed to a state on $\mathcal{B}_\theta$ [Kaschek/Neumaier/Waldmann 08]
- Here: Consider only translationally invariant states ω, i.e.

$$\omega \circ \beta_x = \omega , \qquad x \in \mathbb{R}^d .$$

- QFT examples: Vacuum states, KMS states

Let ω be a β-invariant state on $\mathcal{B}$, and $(\mathcal{H}, \Omega, \pi)$ the GNS data of $(\mathcal{B}, \omega)$, with unitaries $U(x)$ implementing β_x on $\mathcal{H}$. Then

- ω is also a state on $\mathcal{B}_\theta^\infty$, and

$$\omega(A \times_\theta B) = \omega(AB)\,, \qquad A, B \in \mathcal{B}^\infty\,.$$

- The GNS triple $(\mathcal{H}_\theta, \Omega_\theta, \pi_\theta)$ of $(\mathcal{B}_\theta^\infty, \omega)$ is

$$\mathcal{H}_\theta = \mathcal{H}\,, \qquad \Omega_\theta = \Omega\,,$$

$$\pi_\theta(A)\pi(B)\Omega = \pi(A \times_\theta B)\Omega$$

$$= (2\pi)^{-d} \int dp \int dx\, e^{-ipx}\, U(\theta p)\pi(A)U(-\theta p + x)\pi(B)\Omega$$

In particular, $\pi_\theta(A)\Omega = \pi(A \times_\theta 1)\Omega = \pi(A)\Omega$.

Warped Convolutions

- The formula

$$F_\theta\Psi := (2\pi)^{-d}\int dp\int dx\, e^{-ipx}\, U(\theta p)FU(x-\theta p)\Psi$$

makes sense for any smooth $F \in \mathcal{B}(\mathcal{H})^\infty$ on smooth vectors Ψ.

- With spectral resolution $U(x) = \int dE(k)\, e^{ikx}$,

$$F_\theta = (2\pi)^{-d}\int dp\int dx\, e^{-ipx}\, U(\theta p)FU(-\theta p)\int dE(k)\, e^{ikx}$$
$$= \int U(\theta k)FU(-\theta k)\, dE(k)$$

warped convolution deformation [Buchholz/Summers 08]

- Important effect of state/representation: p-integration in Rieffel integral runs only over the spectrum

Warped Convolutions

- The formula

$$F_\theta\Psi := (2\pi)^{-d}\int_S dp\int dx\, e^{-ipx}\, U(\theta p)FU(x-\theta p)\Psi$$

makes sense for any smooth $F\in\mathcal{B}(\mathcal{H})^\infty$ on smooth vectors Ψ.

- With spectral resolution $U(x)=\int_S dE(k)\,e^{ikx}$,

$$F_\theta = (2\pi)^{-d}\int_S dp\int dx\, e^{-ipx}\, U(\theta p)FU(-\theta p)\int_S dE(k)\,e^{ikx}$$
$$= \int_S U(\theta k)FU(-\theta k)\,dE(k)$$

warped convolution deformation [Buchholz/Summers 08]

- Important effect of state/representation: p-integration in Rieffel integral runs only over the spectrum S

Application of Rieffel Deformations to QFT

- Consider a wedge algebra $\mathcal{A} \subset \mathcal{B}$, α, i.e.

$$\alpha_{x,\Lambda}(\mathcal{A}) \subset \mathcal{A} \qquad \text{for } (x,\Lambda) \text{ with } \Lambda W_R + x \subset W_R$$
$$\alpha_{x,\Lambda}(\mathcal{A}) \subset \mathcal{A}' \qquad \text{for } (x,\Lambda) \text{ with } \Lambda W_R + x \subset W_R'$$

- Rieffel's deformation can be applied to $\mathcal{B}$ with action $\beta := \alpha|_{\mathbb{R}^d}$.
- Consider deformed wedge algebra $\mathcal{A}^\theta$ generated by

$$A_1 \times_\theta \dots \times_\theta A_n \,, \qquad A_1, \dots, A_n \in \mathcal{A}^\infty$$

- Lorentz transformations act according to

$$\alpha_{x,\Lambda}(A \times_\theta B) = \alpha_{x,\Lambda}(A) \times_{\Lambda\theta\Lambda^T} \alpha_{x,\Lambda}(B)$$

- To satisfy the isotony condition, need $\Lambda\theta\Lambda^T = \theta$ for $\Lambda W_R \subset W_R$.

Lemma [Grosse/GL 07]

Let for $d = 4$ and $d \neq 4$, respectively,

$$\theta := \begin{pmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa' \\ 0 & 0 & -\kappa' & 0 \end{pmatrix}, \quad \theta := \begin{pmatrix} 0 & \kappa & 0 & \cdots & 0 \\ -\kappa & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

($\kappa, \kappa' \in \mathbb{R}$ free parameters.) Then

- $\Lambda W_R \subset W_R \iff \Lambda\theta\Lambda^T = \theta$,
- $\Lambda W_R \subset W_R' \iff \Lambda\theta\Lambda^T = -\theta$.

- With θ chosen as above, the isotony condition is satisfied for the deformed system $\mathcal{A}^\theta \subset \mathcal{B}$, α.
- For locality condition, need to consider expressions like

$$A \times_\theta (B \times_{-\theta} C) - B \times_{-\theta} (A \times_\theta C)$$

- Compute

$$A \times_\theta (B \times_{-\theta} C) - B \times_{-\theta} (A \times_\theta C)$$
$$= (2\pi)^{-d} \int dp \int dx\, e^{-ipx}\, \alpha_{x/2}\left([\alpha_{\theta p}(A),\, \alpha_{-\theta p}(B)]\right) \alpha_x(C)$$

- In GNS-representation w.r.t. translationally invariant state ω:

$$[\pi(A)_\theta, \pi(B)_{-\theta}]\pi(C)\Omega$$
$$= (2\pi)^{-d} \int_S dp \int dx\, e^{-ipx}\, U(\tfrac{x}{2})\pi([\alpha_{\theta p}(A),\, \alpha_{-\theta p}(B)])U(\tfrac{x}{2})\pi(C)\Omega$$

$\Longrightarrow [\pi(A)_\theta, \pi(B)_{-\theta}] = 0$ if $[\alpha_{\theta p}(A),\, \alpha_{-\theta p}(B)] = 0$ for all $p \in S$.

- If $\kappa \geq 0$, this condition is satisfied for a vacuum state since

$$\theta S \subset \theta \overline{V^+} \subset W_R$$

[Buchholz/Summers 08]

- For this choice of θ, get deformed wedge algebra (in vac. rep.)

$$\pi(\mathcal{A}^\theta) \subset \mathcal{B}(\mathcal{H}),\ \mathrm{ad}U$$

Properties of the deformed theory

- Deformed wedge algebra defines a covariant, local net

$$O \mapsto \pi(\mathcal{A}_\theta(O))$$

  in vacuum representation.

- For a decent energy-momentum spectrum, the two-particle S-matrix can be computed: Use
  1. Reeh-Schlieder for deformed wedge algebras ($\pi_\theta(A)\Omega = \pi(A)\Omega$)
  2. Haag-Ruelle scattering theory for wedge-local operators [Borchers/Buchholz/Schroer 01]

- The S-matrix changes under the deformation

$$\,^{\theta}_{\text{out}}\langle p, q \,|\, p', q' \rangle^{\theta}_{\text{in}} = e^{i|p\theta q|} e^{i|p'\theta q'|} \cdot \,^{0}_{\text{out}}\langle p, q \,|\, p', q' \rangle^{0}_{\text{in}}$$

  [Grosse/GL 07] for deformation of free theory,

  [Buchholz/Summers 08] general case

- S-matrix not Lorentz-invariant

Other examples of deformations

- Consider the Borchers-Uhlmann algebra $\underline{\mathcal{S}}$ over $\mathbb{R}^d$ and a function $R : \{z \in \mathbb{C} \,:\, \operatorname{Im} z \geq 0\} \to \mathbb{C}$ satisfying a number of analyticity and symmetry conditions.
- Pick $\theta \in \mathbb{R}_-^{d\times d}$ as before.
- Define a new product on $\underline{\mathcal{S}}$,

$$\left(\widetilde{f \otimes_\theta^R g}\right)_n(p_1, \dots, p_n) := \sum_{k=0}^{n} \tilde{f}_k(p_1, \dots, p_k)\tilde{g}_{n-k}(p_{k+1}, \dots, p_n) \prod_{l=1}^{k} \prod_{r=k+1}^{n} R(p_l \theta p_r)$$

- $(f \otimes_\theta^R g)^* = g^* \otimes_\theta^R f^*$ and $f \otimes_\theta^R 1 = 1 \otimes_\theta^R f = f$ because of properties of R.
- For $R(u) = e^{iu}$ same as Rieffel deformation

Other examples of deformations

- The Wightman state ω_0 corresponding to the free massive field satisfies
$$\omega_0(f \otimes_\theta^R g) = \omega_0(f \otimes g)\,, \qquad f, g \in \underline{\mathcal{S}}\,.$$
$\longrightarrow$ same structure as before.
- Deformed field operators ϕ_θ^R on undeformed Hilbert space $\mathcal{H}$, generate polynomial algebras $\mathcal{P}_\theta^R(W)$.
- Wedge-locality $[\phi_\theta^R(f), \phi_{-\theta}^R(g)] = 0$ requires analytic properties of R.
- Function R appears in deformation of S-matrix elements.

Local Observables

- The net of deformed wedge algebras determines maximal local (double cone) algebras by intersection
- In $d > 2$, the breaking of Lorentz invariance of the S-matrix implies that the Reeh-Schlieder property for double algebras must be violated.
- $\longrightarrow$ Need more general types of deformations to overcome this
- In $d = 2$, the S-matrix is Lorentz invariant. For an infinite family of functions R, the deformed local net satisfies Reeh-Schlieder, and can be identified with certain integrable models with factorizing S-matrix [Buchholz/GL 04, GL 08]

Conclusion & Outlook

- Deformations of wedge algebras provide a new perspective on the problem of constructing interacting QFTs
- Best studied example in $d = 4$: Rieffel deformations with invariant state $\rightarrow$ lead to wedge-local theories with non-trivial S-matrix
-
- Application of Rieffel techniques to other translationally invariant states (KMS states) and/or curved spacetimes [Morfa-Morales, work in progress]
- More examples of deformations of (at least) free wedge algebras exist [GL, work in progress]
- Construction of QFT on locally noncommutative spacetimes [Waldmann/GL, work in progress]