QFT in curved spacetimes containing null-like boundaries and bulk to boundary correspondence

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Summary

1. Motivation, strategies and general results.
2. Spacetimes asymptotically flat at null infinity.
3. Cosmological models of expanding universes.
4. The Unruh state and the Hadamard property.
5. The double cone and the modular group for the KG field.
6. Open issues.

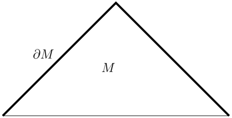
References

- *C.Dappiaggi, V.M., N.Pinamonti,* **RMP 18, 349 (2006)** ,
- *V.M.,* **CMP. 268, 727 (2006)**,
- *V.M.,* **CMP. 279, 31 (2008)**,
- *C.Dappiaggi, V.M., N. Pinamonti:* **CMP. 285, 1129 (2009)**,
- *C.Dappiaggi, V.M., N. Pinamonti:* **JMP. 50, 062304 (2009)**,
- *C.Dappiaggi, V.M., N. Pinamonti,* **arXiv:0907.1034 [gr-qc]**,
- *R.Brunetti, V.M., work in progress.*

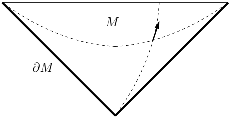
1.1 Motivations and general results.

General motivation: to study both how the (asymptotic) geometry of certain classes of spacetimes selects distinguished Hadamard states for (linear) QFT and general properties of those states.

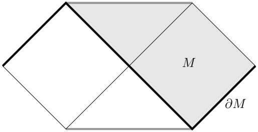
General geometric structure spacetimes in those classes:
spacetime M + light-like (part of) boundary ∂M



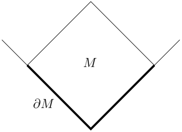
Asypt.flat spacetime at null infinity



Expanding spacetime with past horizon



(extended) Schwarzschild spacetime



Double cone in $\mathbb{M}^4$

1.2 Strategies and general results: geometry.

- $\partial M \simeq \mathbb{R} \times \mathbb{S}^2$ or **unions** of several $\mathbb{R} \times \mathbb{S}^2$ with metric

$$-2dU\,dV + d\theta^2 + \sin^2\theta d\phi^2 \quad \text{where} \quad V = 0\,,$$

- U, V, θ, ϕ coordinates around ∂M (corresp. to $V = 0$),
- $U \in (-\infty, +\infty)$ (affine) parameter of the null geodesics.

- In the asympt. flat and cosmological cases, ∂M admits a distinguished group of diffeomorphisms $\mathcal{G} \ni g : \partial M \to \partial M$;

- ∂M and $\mathcal{G}$ are **universal**: the same for all bulks M matching ∂M ($\Longrightarrow$ $\mathcal{G}$ is ∞-dim. (non-locally-compact Lie) group.)

- $\mathcal{G}$ includes a group $\mathcal{G}_M$ of **Killing isometries** of **every** M matching ∂M: $\exists$ one-to-one homomorphism $\quad h_M : \mathcal{G}_M \to \mathcal{G}$.

- Geometry of ∂M $\Longrightarrow$ symplectic space $(S_{\partial M}, \sigma_{\partial M})$:
- $S_{\partial M} \supset C_0^\infty(\partial M; \mathbb{R})$ real vector space
- $\sigma_{\partial M}(\psi, \psi') \doteq \int_{\partial M} (\psi \partial_U \psi' - \psi' \partial_U \psi) \quad dU \wedge d\mu_{\mathbb{S}^2}$

$\Longrightarrow$ $\exists$ Weyl C^*-algebra $\mathcal{W}(\partial M)$ associated with $(S_{\partial M}, \sigma_{\partial M})$.
Generators $W_{\partial M}(\psi)$ satisfying Weyl **CCR**.

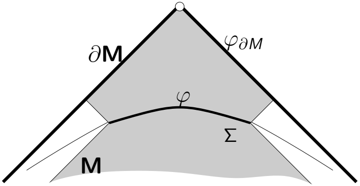
- $S_{\partial M}$ and $\sigma_{\partial M}$ **invariant under** $\mathcal{G}$: $\mathcal{G} \ni g \mapsto \beta_g : S_{\partial M} \to S_{\partial M}$ symplect isomorphisms.

$\Longrightarrow$ $\exists$ rep. of $\mathcal{G} \ni g \mapsto \alpha_g : \mathcal{W}(\partial M) \to \mathcal{W}(\partial M)$ $*$-automorphisms, individuated by $\alpha_g(W_{\partial M}(\psi)) \doteq W_{\partial M}(\beta_g(\psi))$.

- What about the interplay of $\mathcal{W}(\partial M)$ and the **field-observables algebra** $\mathcal{W}(M)$ of any M matching ∂M?

1.4 Strategies and general results: algebras.

- M spacetime matching ∂M, $\mathcal{W}(M)$ **CCR algebra** of a scalar Klein-Gordon field φ. $\mathcal{W}(M)$ associated with (S_M, σ_M):
- S_M space of smooth KG solutions, compactly supp. Cauchy data
- $\sigma_M(\varphi, \varphi') \doteq \int_S (\varphi \nabla_n \varphi' - \varphi \nabla_n \varphi')\, d\mu_\Sigma$



- If $\varphi \in S(M)$ extends to $\varphi_{\partial M} \in S(\partial M)$, Poincaré theorem $\Longrightarrow$

$$\sigma_M(\varphi, \varphi') = \sigma_{\partial M}(\varphi_{\partial M}, \varphi'_{\partial M})$$

Actually not so straightforward (information may escape from the tip of the cone...): to be examined case by case.

1.5 Strategies and general results: algebras.

- If $\Gamma_M : \mathcal{W}(M) \ni \varphi \mapsto \varphi_{\partial M} \in \mathcal{W}(\partial M)$ (linear) exists with $\sigma_M(\varphi, \varphi') = \sigma_{\partial M}(\varphi_{\partial M}, \varphi'_{\partial M}) \implies \Gamma_M$ is **injective** since σ_M nondegenerate.

$\implies \exists!$ $*$-**algebra homomorphism** $\imath_M : \mathcal{W}(M) \to \mathcal{W}(\partial M)$ with $\imath_M(W_M(\varphi)) \doteq W_{\partial M}(\varphi_{\partial M})$, $W_M(\varphi) \in \mathcal{W}(M)$ Weyl generator.

- $\imath_M$ **induces** a state ω_M on **each** $\mathcal{W}(M)$ if a state ω on $\mathcal{W}(\partial M)$ is given

$$\omega_M(a) \doteq \omega_{\partial M}\left(\imath_M(a)\right) \quad \forall a \in \mathcal{W}(M)\,.$$

- It would be nice fixing $\omega_{\partial M}$ such that, for each M:

(1) ω_M is **invariant** under all the Killing symmetries (if any) of M.

(2) ω_M has **positive energy** with respect to every **globally timelike Killing symmetry** of every M,

(3) ω_M is of **Hadamard type**,

(4) ω_M coincides with known states when M is "well known" (e.g. Minkowski vacuum if M is Minkowski spacetime, Bunch-Davies vacuum in deSitter spacetime, Unruh state if M is the extended Schwarzschild space).

- If $\omega_{\partial M}$ is $\mathcal{G}$-invariant and $\imath_M$ and $h_M : \mathcal{G}_M \to \mathcal{G}$ "commute" $\Longrightarrow$ (1) holds.

$$\omega_M(\beta_g^{(M)}(a)) \doteq \omega_{\partial M}(\imath_M(\beta_g^{(M)}(a))) = \omega_{\partial M}(\beta^{(\partial M)}_{h_M(g)}\imath_M(a)) = \omega_{\partial M}(\imath_M(a)) \doteq \omega_M(a)$$

1.7 Strategies and general results: states.

Central question: Are there $\mathcal{G}$-**invariant** states on $\mathcal{W}(\partial M)$?

- Quasifree state $\omega_{\partial M}$ on $\mathcal{W}(\partial M)$ with two-point function on $C_0^\infty(\partial M) \times C_0^\infty(\partial M)$ [Sewell82], [DimockKay87], [KayWald91]:

$$\omega_{\partial M}(\psi, \psi') = -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\psi(U,\omega)\psi'(U',\omega)}{(U-U'-i0^+)^2} dU dU' d\mu_{\mathbb{S}^2}(\omega)$$

(It has to be extended to $S_{\partial M} \times S_{\partial M}$) $\Longrightarrow$

- $\omega_{\partial M}$ well defined ($\exists$ extension to $S_{\partial M}$ ...).
- $\omega_{\partial M}$ $\mathcal{G}$-invariant (a.f. spacetimes and cosmological models).
- $\omega_{\partial M}$ admits **positive energy** w.r.t. the symmetries in $\mathcal{G}$ arising by **timelike Killing vectors** of any bulk M
- That **positive energy**-property **uniquely individuates** $\omega_{\partial M}$,
- If ω_M exists, it is **invariant under the Killing symmetries** of M

Hadamard property of ω_M:

- 2-point function of ω_M on $C_0^\infty(M) \times C_0^\infty(M)$: composition of $T \doteq (U - U' - i0^+)^{-2}\delta(\omega, \omega') \in \mathcal{D}'(\partial M \times \partial M)$ and two **causal propagators** $E_M : C_0^\infty(M) \to C^\infty(\partial M)$ (restricted to ∂M).

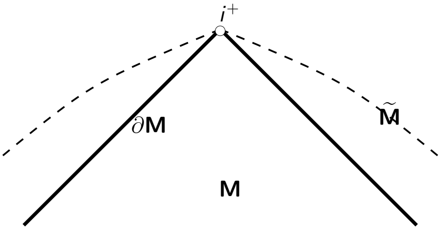
- From thms on composition of WF and propagation of singularities, $WF(E_M)$, $WF(T)$ being known. $\Longrightarrow$

$\omega_M \in \mathcal{D}'(M \times M)$ and $WF(\omega_M)$ satifies the μ **spect.condition**

provided $sing.supp(E_M)$ is controlled near critical "points" (the tip of the cone) to get rid of infrared singularities.

In this case, the μ spect.condition implies that ω_M is Hadamard.

2.1 Spacetimes asymptotically flat at null infinity.



- Vacuum Einstein spacetimes (M, g) "tending to flat spacetimes" at **(future) null infinity** $\Im^+ \doteq \partial M \simeq \mathbb{R} \times \mathbb{S}^2$ ([Wald84] for details)

- $\Im^+ = \partial M$ boundary of M in a larger (nonphysical) spacetime $(\widetilde{M}, \widetilde{g})$. $\widetilde{g} = V^2 g$. $V\restriction_{\partial M} \equiv 0$. (M, g) fulfils Einstein vacuum eq.s about $\Im^+$. $\quad \widetilde{g}|_{\partial M} = -2dUdV + d\theta^2 + \sin^2\theta d\phi^2$

2.2 Spacetimes asymptotically flat at null infinity.

- $(\widetilde{M}, \widetilde{g})$ **not completely determined** by (M, g) $\Rightarrow$ geometry of $\partial M = \Im^+$ fixed up to a group $\mathcal{G}$ of diffeomorphisms: the Bondi Metzner Sachs group $\mathcal{G} \simeq SO(1,3)\uparrow \ltimes C^\infty(\mathbb{S}^2)$.

- [Geroch, Ashtekar, Xanthopoulos ∼80] If $\mathcal{G}_M$ group of Killing isometries of M, $\exists h_M : \mathcal{G}_M \to \mathcal{G}$ injective group homom. (obtained extending M-Killing vectors to $\Im^+$).

- $\mathcal{W}(\partial M)$ and the BMS-invariant state $\omega_{\partial M}$ well defined.

- We consider **massless conformally coupled** fields in M and define (if possible) $\Gamma_M : \varphi_{\partial M} \doteq \lim_{\to \partial M} V^{-1}\varphi \qquad (\widetilde{g} = V^2 g)$.

Problem: Finding sufficient conditions for globally hyperbolic asympt. flat spacetimes (M, g) to define ω_M form $\omega_{\partial M}$.

2.3 Spacetimes asymptotically flat at null infinity.

Sufficient conditions: $(\widetilde{M}, \widetilde{g})$ **globally hyperbolic AND** (M, g) admits **time-like future infinity** i^+ [Friedrich86] (it controls $sing.supp(E_M)$ in particular).

$\Longrightarrow \imath_M : \mathcal{W}(M) \to \mathcal{W}(\partial M)$ is well defined and

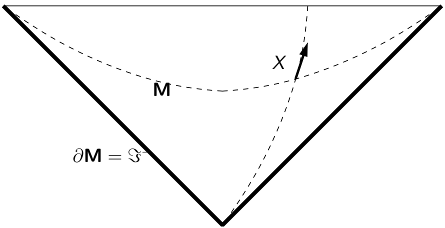
(1) $\omega_M \doteq \omega_{\partial M} \circ \imath_M$ is $\mathcal{G}_M$-**invariant**,

(2) ω_M has **positive-energy** with respect to timelike Killing symmetries of M (if any),

(3) ω_M is **Hadamard**,

(4) ω_M **is the standard Minkowski vacuum** if (M, g) is Minkowski spacetime.

3.1 Cosmological models of expanding universes.



- "Expanding universes" (M, g) with **past cosmological horizon** $\Im^- \simeq \mathbb{R} \times \mathbb{S}^2$. E.g. inflative FRW models perturbations of dS expanding region, **homogeneity and isotropy not necessary**.

- $\Im^- = \partial M$. X timelike conformal Killing vect. light-like on $\Im^-$ (in dS, $X = \partial_\tau$, τ conformal time). X: galaxies worldlines, 3-surfaces $\perp$ to X: co-moving frame.

- $g|_{\partial M} = -2dUdV + d\theta^2 + \sin^2\theta d\phi^2$ $\quad U \in \mathbb{R}$ geodesical affine parameter, ∂M at $V = 0$.

3.2 Cosmological models of expanding universes.

- $\mathcal{G}_M$ subgroup of Killing isometries of M (if any) which become tangent to ∂M approaching there.

- A diffeom. group of ∂M, $\mathcal{G} \simeq C^\infty(\mathbb{S}^2) \ltimes SO(3) \ltimes C^\infty(\mathbb{S}^2)$ exists such that (like BMS), if M matches ∂M, $\exists h_M : \mathcal{G}_M \to \mathcal{G}$ injective group homom. $\omega_{\partial M}$ is $\mathcal{G}$ invariant.

- Consider **generally massive ξ-coupled** fields in M. Define (if possible) $\Gamma_M : \varphi_{\partial M} \doteq \lim_{\to \partial M} \varphi$.

Problem: Finding sufficient conditions on (M, g) for the existence of $\imath_M$ and ω_M .

Sufficient hypotheses: (M, g) belongs to a class of suitable globally hyperbolic FRW perturbations of dS spacetime.

- $\Longrightarrow \exists \imath_M : \mathcal{W}(M) \to \mathcal{W}(\partial M)$ and:

(1) $\omega_M \doteq \omega_{\partial M} \circ \imath_M$ is $\mathcal{G}_M$-**invariant**,

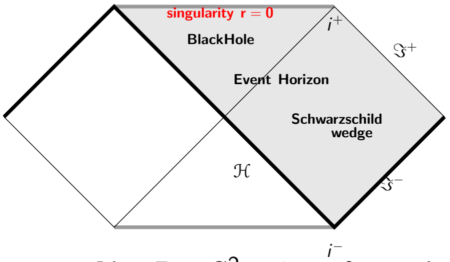
(2) ω_M has **positive-energy** w.r.t. the conformal Killing time M,

(3) ω_M is **Hadamard**,

(4) ω_M coincides with the Bunch-Davies vacuum if $(M, g) = dS$.

(5) ω_M has the properties as those used in cosmology to model scalar fluctuations in the CMB.

- Hadamard prop. established proving that $\omega_M(\cdot, \cdot)$ is the limit (Hörmander top.) of a sequence of distributions with suitable WF. i^- cannot be "added" for $m > 0$.



$M =$ **Schw. wedge** $\cup$ **ev. horizon** $\cup$ **black hole**

$\partial M = \mathcal{H} \cup \Im^-$

- $\mathcal{H} \simeq \mathbb{R} \times \mathbb{S}^2$ union of complete null geodesics, affine par. $U \in \mathbb{R}$ $g\restriction_{\mathcal{H}} = r_S^2(-2dUd\Omega + d\theta^2 + \sin^2\theta d\phi^2)$.
- $\Im^- \simeq \mathbb{R} \times \mathbb{S}^2$ union of complete null geodesics of $\widetilde{g} = g/r^2$, affine par. $v \in \mathbb{R}$ $\widetilde{g}\restriction_{\Im^-} = -2dvd\Omega + d\theta^2 + \sin^2\theta d\phi^2$
- $S_{\partial M} \doteq S_{\mathcal{H}} \oplus S_{\Im^-}$, $\sigma_{\partial M} \doteq \sigma_{\mathcal{H}} \oplus \sigma_{\Im^+}$, and $\omega_{\partial M} \doteq \omega_{\mathcal{H}}^{(U)} \otimes \omega_{\Im^-}^{(v)}$ on $\mathcal{W}(\mathcal{H}) \otimes \mathcal{W}(\Im^-)$
- $S_{\partial M}$ and $S_{\Im^-}$ contain restrictions to $\mathcal{H}$ and $\Im^-$ of φ and $r\varphi$ with $\varphi \in S(M)$ (space of solutions of massless KG equation).

4.2 Schwarzschild spacetime and the Unruh state.

- Estimate of $\varphi\restriction_{\mathcal{H}}$ and $r\varphi\restriction_{\Im^-}$ known [DafermosRodnianski09]: slow decay. Extension of KW two-point function to $S_{\partial M}$ laborious, local Sobolev extensions used.

- Injective $*$-homomorphism $\imath_M : \mathcal{W}(M) \to \mathcal{W}(\mathcal{H}) \otimes \mathcal{W}(\Im^-)$ well defined $\Longrightarrow \omega_M \doteq \omega_{\partial M} \circ \imath_M$ well-defined and:

(1) ω_M invariant under all Killing symmetries of M.

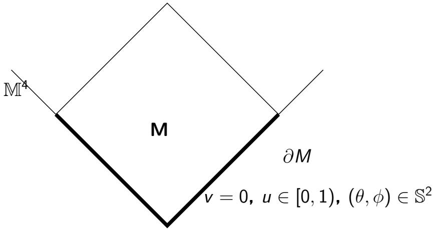
(2) ω_M is everywhere Hadamard on M (static region, event horizon, black hole region). Very laborious proof relying on: (a) properties of passive states [SahlmannVerch00-01] in Schw. region, (b) singularities propagation, (c) WFs composition.

(3) ω_M describes Hawking radiation about $\Im^+$. By direct inspection or by a general result [FredenhagenHaag92] based on the Hadamard property aroud $\mathcal{H}_{ev}$.

- The used procedure to define ω_M and the appearance of Hawking radiation is in agreement with the recipe for constructing and the properties of the Unruh state.

- ω_M can be extended to the whole Kruskal manifold. The extended state cannot be Hadamard on $\mathcal{H}$ due to Kay-Wald uniqueness theorem.

5.1 Double cones in $\mathbb{M}^4$.



- M **double cone** in $\mathbb{M}^4$, $\mathcal{W}_m(M)$ Weyl algebra of KG field with mass $m > 0$. Minkowski vacuum Ω_m, GNS triple $(\mathfrak{H}_{\Omega_M}, \pi_{\Omega_m}, \Psi_{\Omega_m})$.

- Ψ_{Ω_m} cyclic and separating for $\pi_{\Omega_m}(\mathcal{W}_m(M))''$
$\Longrightarrow$ $\pi_{\Omega_m}(\mathcal{W}_m(M))''$ admits **modular group** $\alpha_t^{(m)}(\cdot)$.

- *No* explicit representation for $\alpha_t^{(m)}(\cdot)$ $(m > 0)$ known. [Hislop-Longo82] Known for $m = 0$, conformal techniques reducing to the Rindler wedge. For $m > 0$, indirect representations [Figliolini-Guido89] even for more general regions.

5.2 Double cones in $\mathbb{M}^4$.

- $\imath_M : \mathcal{W}_m(M) \to \mathcal{W}(\partial M)$ and $\omega_{\partial M}$ (independent from m) are well defined. Defining $\omega_M \doteq \omega_{\partial M} \circ \imath_M$, one has:

(1) $\omega_M \equiv \Omega_m \restriction_{\mathcal{W}_m(M)}$ for every $m > 0$.

(2) $\pi_{\Omega_m}(\mathcal{W}_m(M))''$ and $\pi_{\omega_{\partial M}}(\mathcal{W}(\partial M))''$ unitarily equivalent by means of V_m implementing $\imath_M$ ($V_m \pi_{\Omega_m} V_m^* = \pi_{\omega_{\partial M}} \circ \imath_M$) preserving GNS cyclic vectors ($V_m \Psi_{\Omega_m} = \Psi_{\omega_{\partial M}}$) .

$\Longrightarrow \alpha_t^{(m)}(A) = V_m \alpha_t^{(\partial M)} \left(V_m^* A V_m\right) V_m^*$

$\alpha_t^{(\partial M)} =$ mod. group for the theory on ∂M, **independent from** m

5.3 Double cones in $\mathbb{M}^4$.

- $\alpha_t^{(\partial M)}$ explicitly computable and has a geometric interpretation: it is induced by the 1-parameter group of the vector field X on ∂M:

$$X \doteq u(1-u)\partial_u$$

$$u \doteq t + |\mathbf{x}|, \quad v \doteq t - |\mathbf{x}| \quad (\partial V \text{ at } v = 0 \text{ with } u \in (0,1))$$

$\Longrightarrow$ Indirect geometric representation of the modular group $\alpha_t^{(m)}$ of $\pi_{\Omega_m}(\mathcal{W}(M))''$ found. All information on $m > 0$ embodied in V_m.

- Further step (still in progress): explicitly computing the self-adjoint generator of $\alpha_t^{(m)}$, using the fact that V_m implements $\imath_M$, making use of the explicit solution of Goursat problem in M with data on ∂M.

6.1 Open issues.

- Referring to all considered cases. Extension of $\imath_M$ to the algebra of Wick-polynomials, to encompass interactions at perturbative level.

- Asymp. flat spacetimes and Schwarzschild spacetime: to investigate massive fields.

- Existence proof and Hadamard property of the Hartle-Hawking state (even for the massive case).

- Expanding universes, relation between ω_M and adiabatic vacua [Parker-Fulling73,... Lüders-Roberts90, Junker-Schrohe02].

- In the GNS representation of $\omega_{\partial M}$, the Reeh-Schlieder prop. holds. To export this property in the bulk M (GNS representation of ω_M).