Localization in Nets of Standard Spaces

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joint work with ROBERTO LONGO

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ESI

Model building in AQFT

Recently, many new ideas about model building within the setting of AQFT. Partial list:

- construction of interaction-free theories by modular localization [Brunetti/Guido/Longo 2002]
- boundary QFT models [Longo/Rehren 2004]
- Construction of integrable models [Schroer 2000, GL 2003, Buchholz/GL 2004, GL 2006, Bostelmann/Cadamuro 2012,...]
- Models of string-local infinite spin fields [Mund/Schroer/Yngvason 2006]
- construction of conformal local nets by framed VOAs [Kawahigashi/Longo 2006]
- Deformations of QFTs [Grosse/GL 2007, Buchholz/GL/Summers 2011, GL 2012, Plaschke 2013, Alazzawi 2013, GL/Schlemmer/Tanimoto 2013]
- Constructions with endomorphisms of standard pairs [Longo/Witten 2011, Tanimoto 2012, Bischoff/Tanimoto 2013]
- ...

Modular theory and standard spaces

Important mathematical tool: Modular theory.

- For von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ with cyclic and separating vector Ω, the real subspace $H := \overline{\mathcal{A}(\mathcal{O})_{\text{sa}}\Omega} \subset \mathcal{H}$ is standard:

$$\overline{H + iH} = \mathcal{H}\,, \qquad H \cap iH = \{0\}\,.$$

- Modular data of $(\mathcal{A}, \Omega)$ completely encoded in H:

$$S : H + iH \to H + iH, \qquad h + ik \mapsto h - ik\,.$$

- Polar decomposition of S gives interesting data $(\mathcal{J}, \Delta^{it})$. In particular

$$\mathcal{J}H = H' = \text{symplectic complement w.r.t. } \operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle, \qquad \Delta^{it}H = H\,.$$

- "symplectic complement replaces commutant"

Von Neumann algebras and real standard spaces

$$\mathcal{A}(\mathcal{O})$$

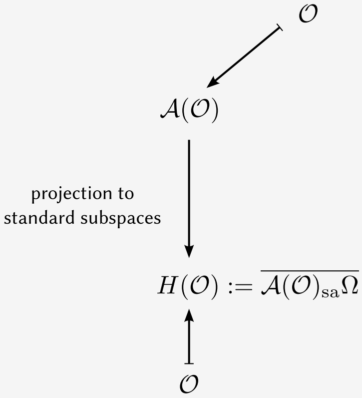
projection to standard subspaces

$$\downarrow$$

$$H(\mathcal{O}) := \overline{\mathcal{A}(\mathcal{O})_{\mathrm{sa}}\Omega}$$

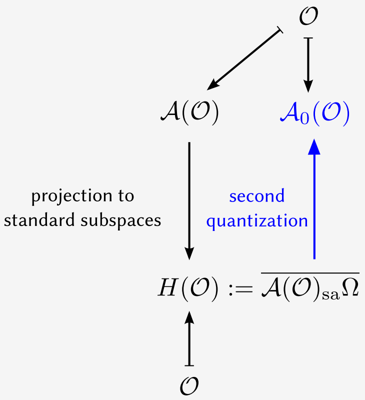
- important data (but not the full algebraic structure) encoded in standard spaces $H(\mathcal{O})$

Von Neumann algebras and real standard spaces



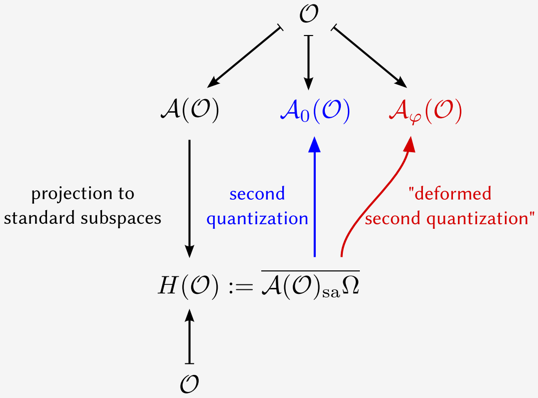
- $\mathcal{O} \mapsto H(\mathcal{O})$ inherits isotony, covariance, locality (with symplectic complements instead of commutants) from $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$

Von Neumann algebras and real standard spaces



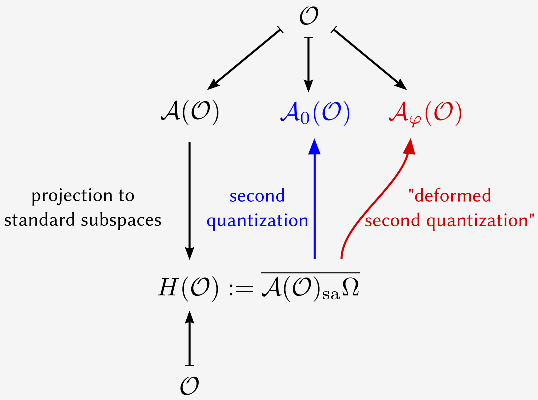
- Can go back to algebraic setting by second quantization, $H(\mathcal{O}) \mapsto \mathcal{A}_0(\mathcal{O}) := \{\mathrm{Weyl}(h) \,:\, h \in H(\mathcal{O})\}''$
- Free field theory $\Leftrightarrow$ net of standard spaces

Von Neumann algebras and real standard spaces



- Also "deformed" versions of second quantization exist; give interacting nets $\mathcal{A}_\varphi$ (φ = 2-particle S-matrix). So far under control for integrable models, see talks by Sabina (today) and Yoh (Friday)

Von Neumann algebras and real standard spaces



- **Focus here:** Nets of standard spaces and their properties
- Simplified version in comparison to von Neumann algebra situation

Standard pairs and nets of standard spaces

Definition

A (1- or 2-dimensional) standard pair (H, T) consists of

- a real standard subspace $H \subset \mathcal{H}$
- a unitary strongly continuous positive energy representation T of the translations such that $T(x)H \subset H$ for x "on the right".

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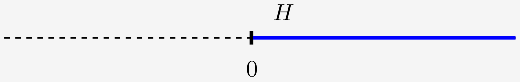

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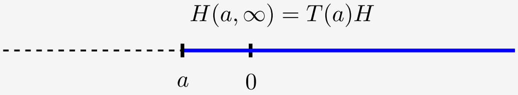

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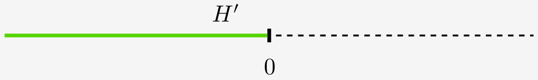

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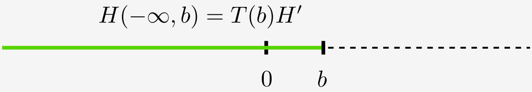

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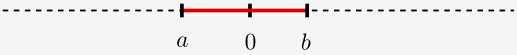

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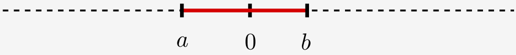

- Gives map $I \longmapsto H(I)$ from intervals in $\mathbb{R}$ to real subspaces of $\mathcal{H}$.
- Same construction can be done in $d = 2$ with the right wedge instead of $\mathbb{R}_+$.

From standard pairs to nets of standard spaces

- $I \mapsto H(I)$ is isotonous, local, T-covariant.
- By Borchers' Theorem, T extends to a (anti-) unitary representation U of the "$(ax+b)$-group" (in $d=1$) or the proper 2d Poincaré group (in $d=2$), under which $I \mapsto H(I)$ is still covariant.

Theorem

If (H, T) is non-degenerate (no non-zero T-invariant vectors), then $H(I)$ is standard for any non-empty interval I.

- follows essentially from [Brunetti/Guido/Longo 2002]
- No comparable result for von Neumann-algebraic case exists.
- The functions φ used in the "deformed second quantization" appear in standard space setting when passing to endomorphisms/subnets.

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Endomorphism Subnets H_V

Definition

An endomorphism of a standard pair (H, T) is a unitary V with

- $VH \subset H$
- $[V, T(x)] = 0$ for all x.

Endomorphisms form semigroup $\mathcal{E}(H, T)$.

Given $V \in \mathcal{E}(H, T)$, define

$$H_V(a, b) := H(-\infty, b) \cap VH(a, \infty)$$

and analogously in $d = 2$.

- Setting $V = 1$ returns previous construction.
- For general endomorphism V, have inclusions (subnet)

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- For general V, have T-covariant local net $I \mapsto H_V(I)$ of real subspaces
- H_V will be fully U-covariant only if $VH = H$.
- Main question: Are the $H_V(I)$ cyclic or at least non-trivial?
- Trivial example: $V = T(x)$, $x \geq 0$, then

$$H_V(I) = \begin{cases} \{0\} & |I| \leq x \\ \text{cyclic} & |I| > x \end{cases}$$

Definition

The minimal localization radius r_V (of the net H_V) is

$$r_V := \inf\{r \geq 0 \,:\, H_V(-r, r) \neq \{0\}\} \in [0, \infty]$$

(no non-zero vectors localized in intervals shorter than $2r_V$.)

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For understanding H_V, one needs to understand V.

Definition

A symmetric inner function on the upper half plane is an analytic bounded function $\varphi : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\varphi(-p) = \overline{\varphi(p)} = \varphi(p)^{-1}\,, \qquad p \geq 0\,.$$

Theorem (Longo/Witten 2011)

There exists a unique 1d non-degenerate standard pair (H, T) with U irreducible. Its endomorphism semigroup is

$$\mathcal{E}(H, T) = \{\varphi(P) \ : \ \varphi \ \textit{ symmetric inner}\,\},$$

where P is the generator of T.

Structure of symmetric inner functions matches that of scattering functions up to one condition.

Inner functions

Canonical Factorization

Any symmetric inner function φ is of the form

$$\varphi(p) = \pm e^{ipx}\, B(p)\, S(p)\,,$$

with

- $x \geq 0$
- B a (symmetric) Blaschke product, $B(p) = \prod_n \frac{p-p_n}{p-\overline{p_n}}$
- S singular inner, $S(p) = e^{-i\int d\mu(t)\,\frac{1+pt}{p-t}}$

$$\varphi \Longleftrightarrow x, \{p_n\}_n, \mu$$

Calculating r_φ

What is the localization radius r_φ of the subnet with $V = \varphi(P)$ and the unique irreducible 1d standard pair?

Localization radii of elementary factors:

inner function φ	localization radius r_φ
$\pm e^{ipx}$	$x/2$
single Blaschke factor	0
singular function	∞

- Need to consider infinite products, but $\varphi \longmapsto r_\varphi$ discontinuous (cf. [Tanimoto 2011] for similar effect)
- important quantity: convergence exponent of the zeros $\{p_n\}$ of φ,

$$\rho_\varphi := \inf\{\alpha \geq 0 \,:\, \sum_n |p_n|^{-\alpha} < \infty\} \in [0, \infty]$$

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Theorem

1. *If* $\rho_\varphi > 1$ *or* $\mu_\varphi \neq 0$, *then* $r_\varphi = \infty$ *(all interval spaces trivial).*
2. *If* $\rho_\varphi < 1$, $\mu_\varphi = 0$, *then* $r_\varphi = \frac{1}{2} x_\varphi$ *(all interv. sp. cyclic if* $x_\varphi = 0$*).*
3. *If* $r > r_\varphi$, *then* $H_\varphi(-r, r)$ *is cyclic.*

- Proof relies on explicit characterization of the spaces $H(-r, r)$ in the (unique) irreducible case:
- In $\mathcal{H} = L^2(\mathbb{R}_+, dp/p)$, a function is localized in $H(-r, r)$ iff it extends to an entire function of exponential type at most r, with $\overline{\psi(-\overline{p})} = \psi(p)$.
- + complex analysis (entire functions, canonical products …)

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For intermediate case $\rho_\varphi = 1$:

Example

$\varphi(p) := \frac{\sin(\nu p - iq)}{\sin(\nu p + iq)}$, $\nu, q > 0$, is a symmetric inner function with

$$x_\varphi = 0, \qquad \mu_\varphi = 0, \qquad \rho_\varphi = 1, \qquad r_\varphi = \nu\,.$$

- Get nets (of subspaces or von Neumann algebras) with intrinsic minimal localization length.
- Regularity of endomorphism (no singular part, zeros not too dense) is necessary (and sufficient) for rich local structure.
- $\rightarrow$ Surprising analogies to integrable models and their scattering functions.

Symmetric inner functions vs. scattering functions

- A symmetric inner function is called a scattering function if it satisfies $\varphi = \gamma(\varphi)$, where $\gamma(\varphi)(p) = \overline{\varphi(1/\bar{p})}$, $\mathrm{Im}p > 0$ (cf. Sabina's talk)
- A scattering function is called regular iff $\varphi \circ \exp$ extends analytically and bounded to $-\varepsilon < \mathrm{Im}\theta < \pi + \varepsilon$ for some $\varepsilon > 0$.
- For regular scattering functions, the inverse scattering problem can be solved by an operator-algebraic construction. Have there $r_\varphi < \infty$ respectively $r_\varphi = 0$ [GL 2006]
- Here: If φ is a scattering function, then either $\rho_\varphi = 0$ or $\rho_\varphi = \infty$. If $\rho_\varphi = 0$, then regularity of φ is equivalent to $r_\varphi = 0$.

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The 2d situation

- In $d = 2$, the non-degenerate irreps U (of the 2d Poincaré group) are uniquely labeled by either a mass $m > 0$, or $m = 0$ and choice of left/right.
- The $m = 0$ irreps give the same nets as in 1d (chiral situation). $\rightarrow$ focus on massive case.
- Generalization of Longo/Witten Theorem to massive 2d case:

Theorem

Let (H, T) be a non-degenerate 2d standard pair with massive multiplicity free representation U. Then

$$\mathcal{E}(H, T) = \{\psi(P_+, M) \,:\, \psi \in L^\infty(\mathbb{R}^2_+),\ \psi(\,\cdot\,, m) \text{ symmetric inner}\}$$

- P_+: generator of lightlike translations, M: mass operator.
- Examples: $U = U_m$ irreducible, or $U = U_m \otimes_+ U_m$ (symmetric tensor square, "2 particle situation"), ...

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The 2d situation - localization radius

- Localization radius $r_{m,\varphi}$ of net $\mathcal{O} \mapsto H^m_\varphi(\mathcal{O})$ with irreducible $U = U_m$ and $V = \varphi(P_+)$?

1. $r > r_{m,\varphi} \Longrightarrow H^m_\varphi(\mathcal{O}_r)$ is cyclic.
2. $\frac{1}{2}\max\{x_\varphi, x_{\gamma(\varphi)}\} \leq r_{m,\varphi} \leq \min\{r_\varphi, r_{\gamma(\varphi)}\}$
3. If $\operatorname{supp}\mu_\varphi \neq \{0\} \Rightarrow r_{m,\varphi} = \infty$.
4. But there also exist Blaschke products φ such that $r_\varphi = r_{\gamma(\varphi)} = \infty$, but $r_{m,\varphi} = 0$.
   (analogous to scaling limits of integrable models, [Bostelmann/GL/Morsella 2011])

- The symmetry $\varphi \mapsto \gamma(\varphi)$ corresponds to time reflection.

Conclusions

- Have studied (sub-)nets of standard spaces and their localization properties.
- Regularity of endomorphism influences localization radius.
- Similarities to integrable models ($\varphi =$ 2-particle S-matrix)
- Link between endomorphism picture and deformation picture not yet clear, to be investigated also at 2-particle level
- In higher particle situations (tensor products of standard subspaces), $\mathcal{E}(H, T)$ will be non-abelian and also contain integral operators (momentum transfer).
- Should provide input into the construction of models with stronger interaction.