Bimodules and non-commutativity in fundamental physics

by John Barrett

C vs NC

space $\rightsquigarrow$ commutative $*$-algebra

vector bundle $\rightsquigarrow$ module over algebra

"NC space" $\rightsquigarrow$ NC $*$-algebra $\mathcal{A}$

"NC vector bundle" $\rightsquigarrow$ bimodule over algebra $\mathcal{H}$

Real structure

$J : \mathcal{H} \to \mathcal{H}$ antilinear

$J^2 = \pm 1$

$J(a\psi b) = b^* (J\psi) a^*$

Examples

- Standard model internal space
- Fuzzy coadjoint orbits
- Fuzzy torus : spin structure, π_1 etc.

SM algebra

$$\mathcal{A} \ni a = \left(\begin{array}{c|c|cc} q & \cdot & \multicolumn{2}{c}{\cdot} \\ \hline \cdot & \begin{matrix} \lambda & \cdot \\ \cdot & \bar{\lambda} \end{matrix} & \multicolumn{2}{c}{\cdot} \\ \hline \multirow{2}{*}{$\cdot$} & \multirow{2}{*}{$\cdot$} & \lambda & \cdot \\ \cline{3-4} & & \cdot & m \end{array}\right)$$

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathbb{H}$$

$$\lambda \in \mathbb{C}$$

$$m \in M_3(\mathbb{C})$$

Gauge group: $\quad G = \{ a \mid a a^* = 1, \ \det a = 1 \}$

$$\rightsquigarrow \quad q \in SU(2), \quad \lambda \in U(1)$$

$$m = \mu s, \quad s \in SU(3), \quad \lambda \mu^3 = 1$$

SM internal space

$$\mathcal{H} \ni \psi = \left(\begin{array}{cccc|cccc} & & & & \nu_L & u_L & u_L & u_L \\ & 0 & & & e_L & d_L & d_L & d_L \\ & & & & \nu_R & u_R & u_R & u_R \\ & & & & e_R & d_R & d_R & d_R \\ \hline \bar{\nu}_L & \bar{e}_L & \bar{\nu}_R & \bar{e}_R & & & & \\ \bar{u}_L & \bar{d}_L & \bar{u}_R & \bar{d}_R & & & 0 & \\ \bar{u}_L & \bar{d}_L & \bar{u}_R & \bar{d}_R & & & & \\ \bar{u}_L & \bar{d}_L & \bar{u}_R & \bar{d}_R & & & & \end{array} \right)$$

bimodule: $a \psi a'$

matrix x.

$\otimes \mathbb{C}^3$ (generations)

Action of $a \in G$: $\psi \longmapsto a \psi a^*$ "adjoint"

$\rightsquigarrow$ SM charges

Coadjoint orbits

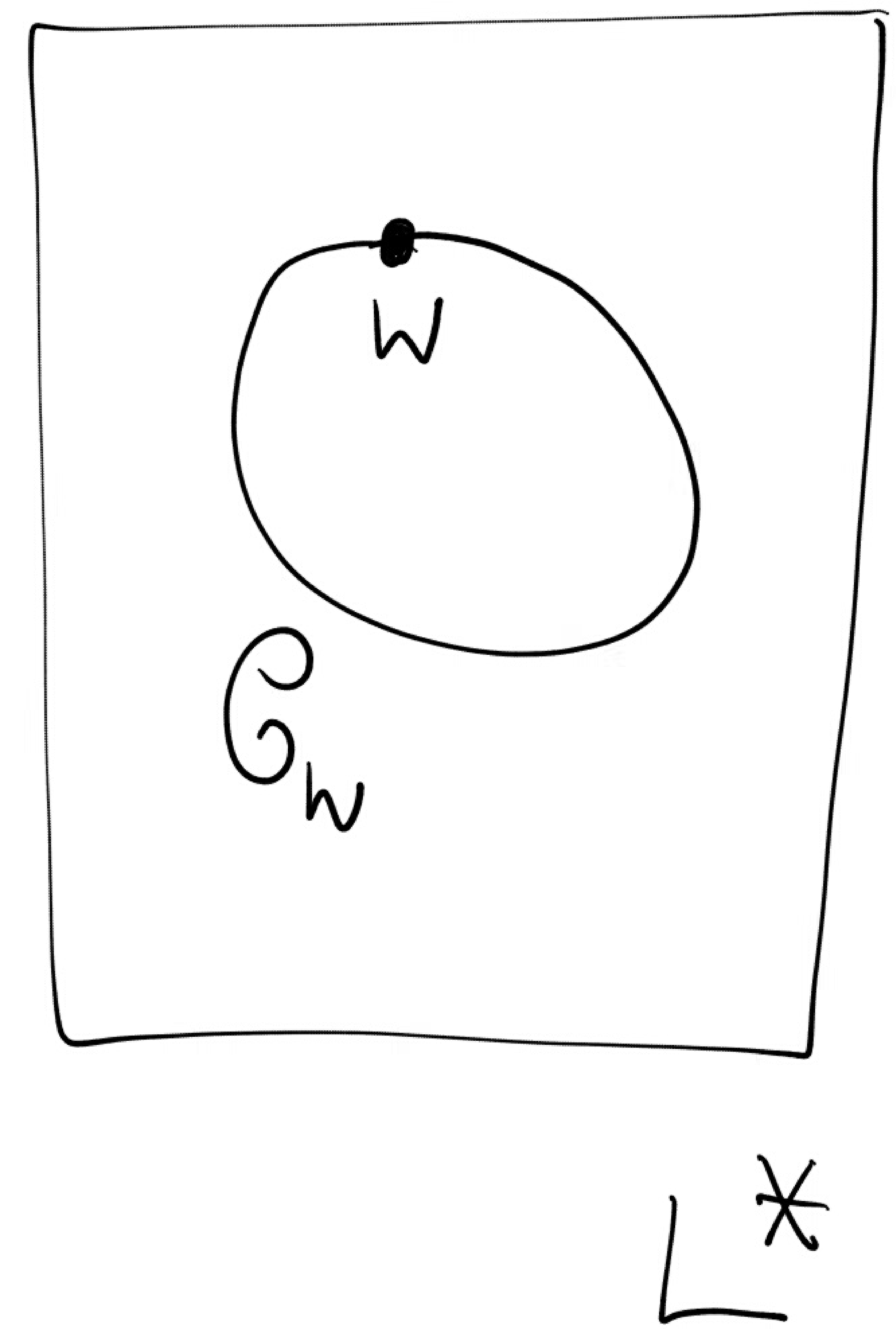
G : Lie group

L : its Lie algebra

$w \in L^*$

$$\mathcal{O}_w = G\{w\} \cong G/H$$

G semisimple $\rightsquigarrow$ $\mathcal{O}_w$ Kähler.



Functions on coadjoint orbit

$L^2(\mathcal{O}_w)$ is a rep of G

If $V \in \mathrm{Irrep}(G)$:

multiplicity of V in $L^2(\mathcal{O}_w)$

$= \dim(H\text{-invariants} \subset V)$

e.g. $G = SU(2)$: $\mathrm{Irrep}(SU(2)) = \{V_n \mid n = 0, 1, 2, \dots\}$

$\mathcal{O}_w \cong S^2 \quad (w \neq 0)$, $\quad L^2(\mathcal{O}_w) \cong \bigoplus_{n=0}^{\infty} V_{2n}$.

Fuzzy sphere

Pick $V_n \in \mathrm{Irrep}(SU(2))$ $\qquad V_n \cong \mathbb{C}^N, \; N = n+1$

$\mathcal{A} = M_N(\mathbb{C})$

Bimodule $\mathcal{H}_N = M_N(\mathbb{C})$ $\quad$ (or $M_N(\mathbb{C}) \otimes \mathbb{C}^2$)

$\cong V_n \otimes V_n^*$ using adjoint action

$\cong V_0 \oplus V_2 \oplus V_4 \oplus \ldots V_{2n}$.

$\mathcal{H}_N \subset \mathcal{H}_{N+1} \subset \ldots \subset L^2(S^2)$

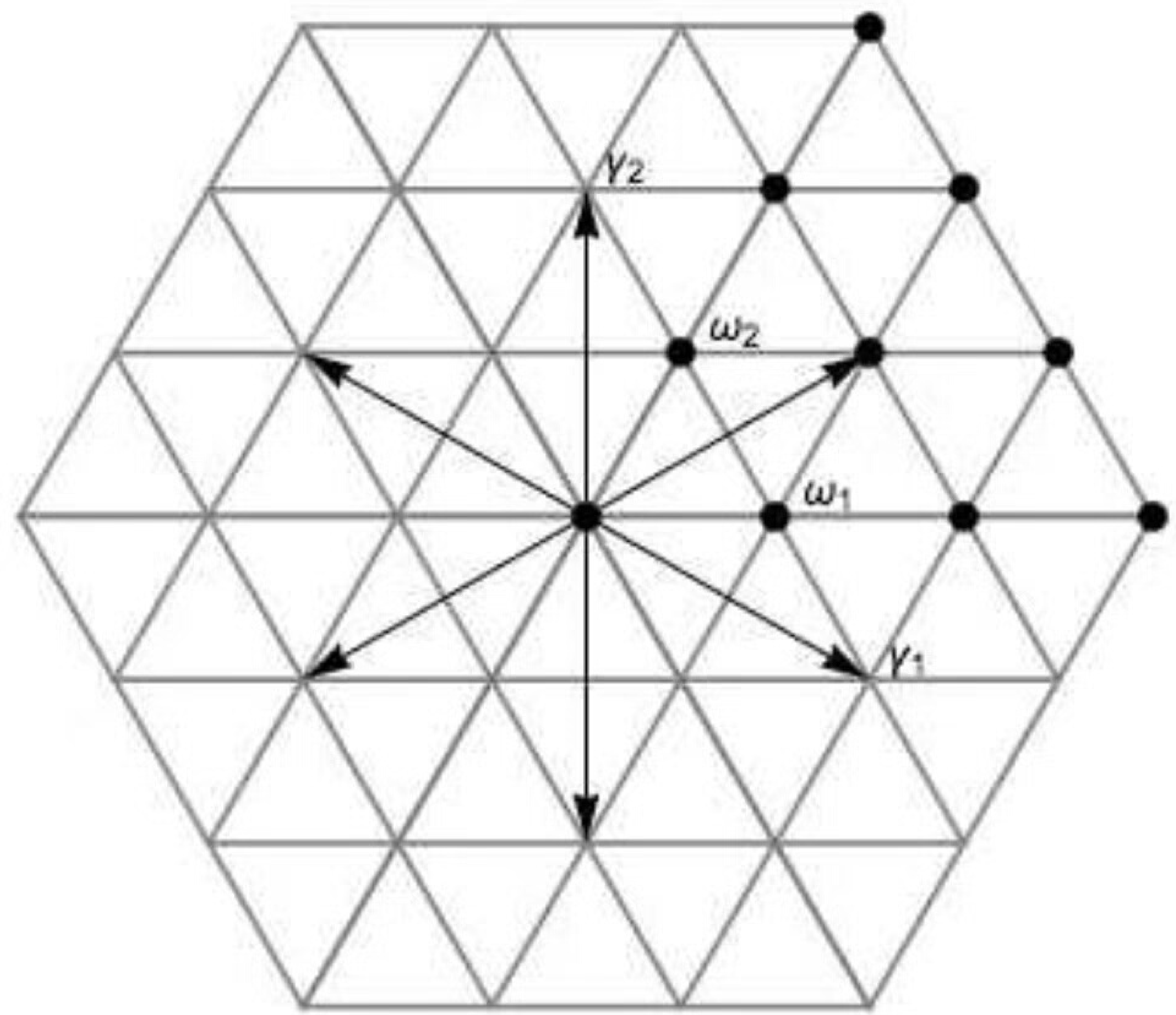
Fuzzy coadjoint orbit

$L^2(\mathcal{O}_w)$ is approximated by $V_w \otimes V_w^*$

In fact, $V_w \otimes V_w^* \subset V_{w'} \otimes V_{w'}^* \subset L^2(\mathcal{O}_{w'})$

if $w' - w \geq 0$

SU(3) weight diagram



$\omega_1 = (1, 0)$
$\omega_2 = (0, 1)$

Algebraically integral elements (triangular lattice), dominant integral elements (black dots), and fundamental weights for sl(3,C) http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/tensor.act?x1=5&x2=3&y1=3&y2=5&rank=2&group=A2

Decomposition of tensor product of [5,3] and [3,5] in A2

Below you find the decomposition of the tensor product of the irreducible A2-modules with highest weights [5,3] and [3,5] into its irreducible factors; the result was computed by LiE using Klimyk's formula. Each term represents a different highest weight of an irreducible module occurring in the decomposition, prefixed by its multiplicity of occurrence. The tensor product has dimension 14400.

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1X[ 8, 8] +1X[ 9, 6] +1X[ 6, 9] +1X[10, 4] +2X[ 7, 7] +1X[ 4,10] +
1X[11, 2] +2X[ 8, 5] +2X[ 5, 8] +1X[ 2,11] +2X[ 9, 3] +3X[ 6, 6] +
2X[ 3, 9] +1X[10, 1] +3X[ 7, 4] +3X[ 4, 7] +1X[ 1,10] +2X[ 8, 2] +
4X[ 5, 5] +2X[ 2, 8] +1X[ 9, 0] +3X[ 6, 3] +3X[ 3, 6] +1X[ 0, 9] +
2X[ 7, 1] +4X[ 4, 4] +2X[ 1, 7] +3X[ 5, 2] +3X[ 2, 5] +1X[ 6, 0] +
4X[ 3, 3] +1X[ 0, 6] +2X[ 4, 1] +2X[ 1, 4] +3X[ 2, 2] +1X[ 3, 0] +
1X[ 0, 3] +2X[ 1, 1] +1X[ 0, 0]

Computation time 0.00 sec.
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agree with $L^2(\mathcal{B}_w)$

If you like, you may look at the implementation that was used (function *simp_tensor_irr* on page 2), which also involves routines for the traversal of Weyl group orbits.

You may go back and try another example.

$\mathcal{B}_w \cong SU(3)/_{\text{max. Torus}}$

Euclidean group $E(2)$

Coadjoint orbits: $(p_1, p_2, \omega) \in L^*$

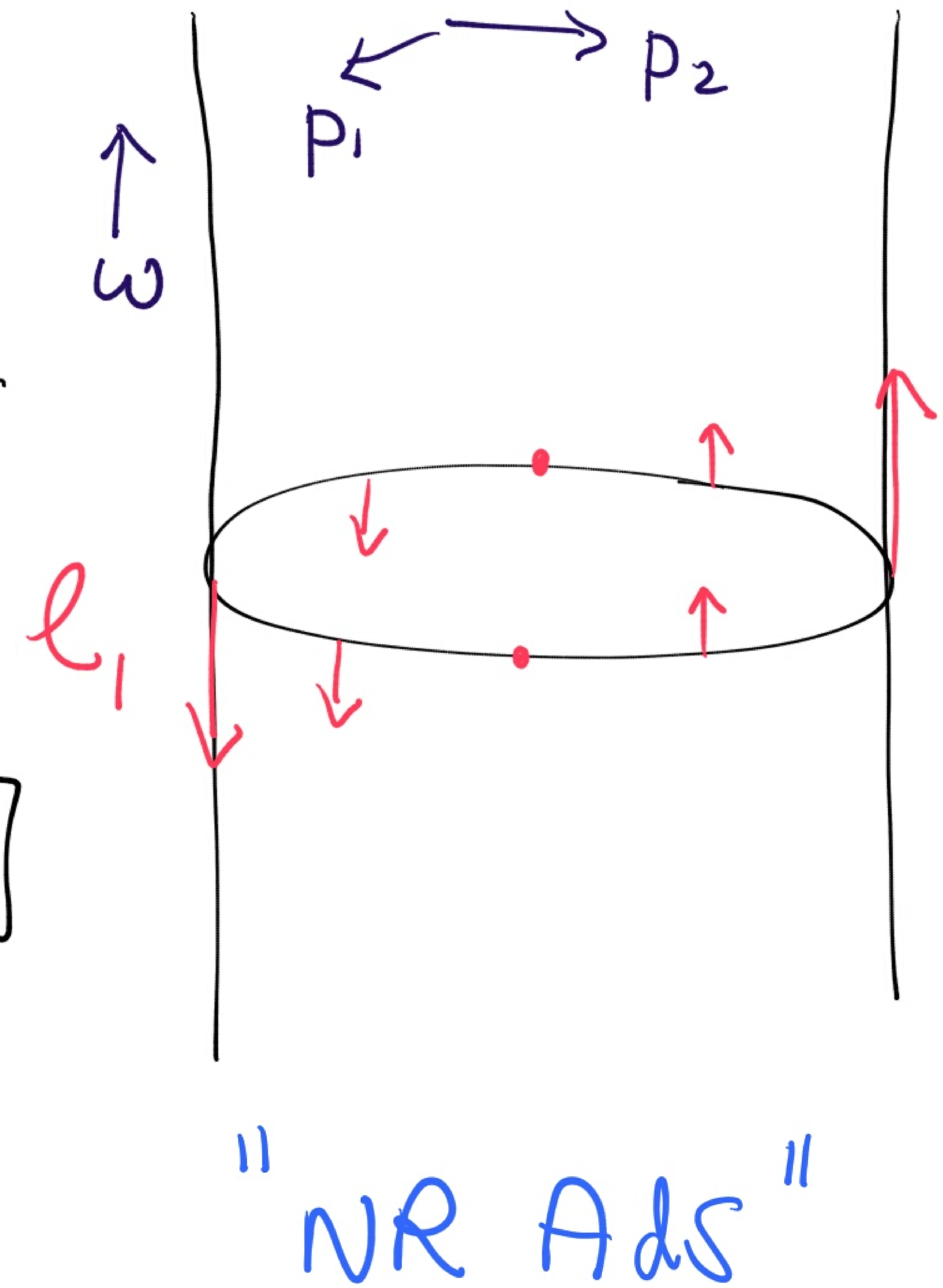
$$p_1^2 + p_2^2 = m^2 \neq 0$$

Laplacian $\Delta_B = -\frac{\partial^2}{\partial \omega^2}$. Symplectic.

Fuzzy cylinder: $\mathcal{H}_m = V_m \otimes V_m^*$

$$\Delta_m \psi = -[\ell_1, [\ell_1, \psi]] - [\ell_2, [\ell_2, \psi]]$$

Spectrum of $\Delta_m = [0, 4m^2]$.



NC covering spaces

$\mathcal{A} \subset \mathcal{B}$ $*$-algebras

Define $\mathcal{G} \subset \mathcal{A}$ central unitaries (abelian group)

$\mathcal{G}$ acts on $\mathcal{B}$: $b \longmapsto g b g^{-1}$ deck transformations

If $\mathcal{H}$ is a bimodule for $\mathcal{B}$, then

$$\mathcal{H} = \bigoplus_{\chi} \mathcal{H}_{\chi} \qquad \chi : G \to U(1)$$

$\mathcal{H}_{\chi}$ are bimodules for $\mathcal{A}$

Order two coverings

S_1, S_2 spin structures

$\rightsquigarrow$ $S_1 - S_2 \in H_1(M, \mathbb{Z}_2)$

M – manifold

$\pi_1(M) \longrightarrow H_1(M; \mathbb{Z}_2)$ $\quad$ $\mathbb{Z}_2$-Hurewicz hom

determines covering $p : \hat{M} \to M$

with deck transformations $G = H_1(M; \mathbb{Z}_2)$

If $\hat{M}$ is compact, spin, $\mathcal{H} = L^2(\hat{M}, \text{spinors})$

then $\mathcal{H}_\chi \cong L^2(M, \text{spinors})$ with all spin structures

Fuzzy torus

$$\mathcal{B} = M_{4N}(\mathbb{C}) = \langle C, S \rangle, \quad J = *$$

$$\text{with } CS = e^{2\pi i/4N} SC \qquad \text{clock, shift}$$

$$C^{4N} = S^{4N} = 1$$

$$\mathcal{A} = \langle U, V \rangle \text{ with } U = C^2, \; V = S^2$$

$$\Rightarrow \quad UV = e^{2\pi i/N} VU$$

$$\mathcal{G} = \langle U^N, V^N \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 .$$

Fuzzy torus bimodules

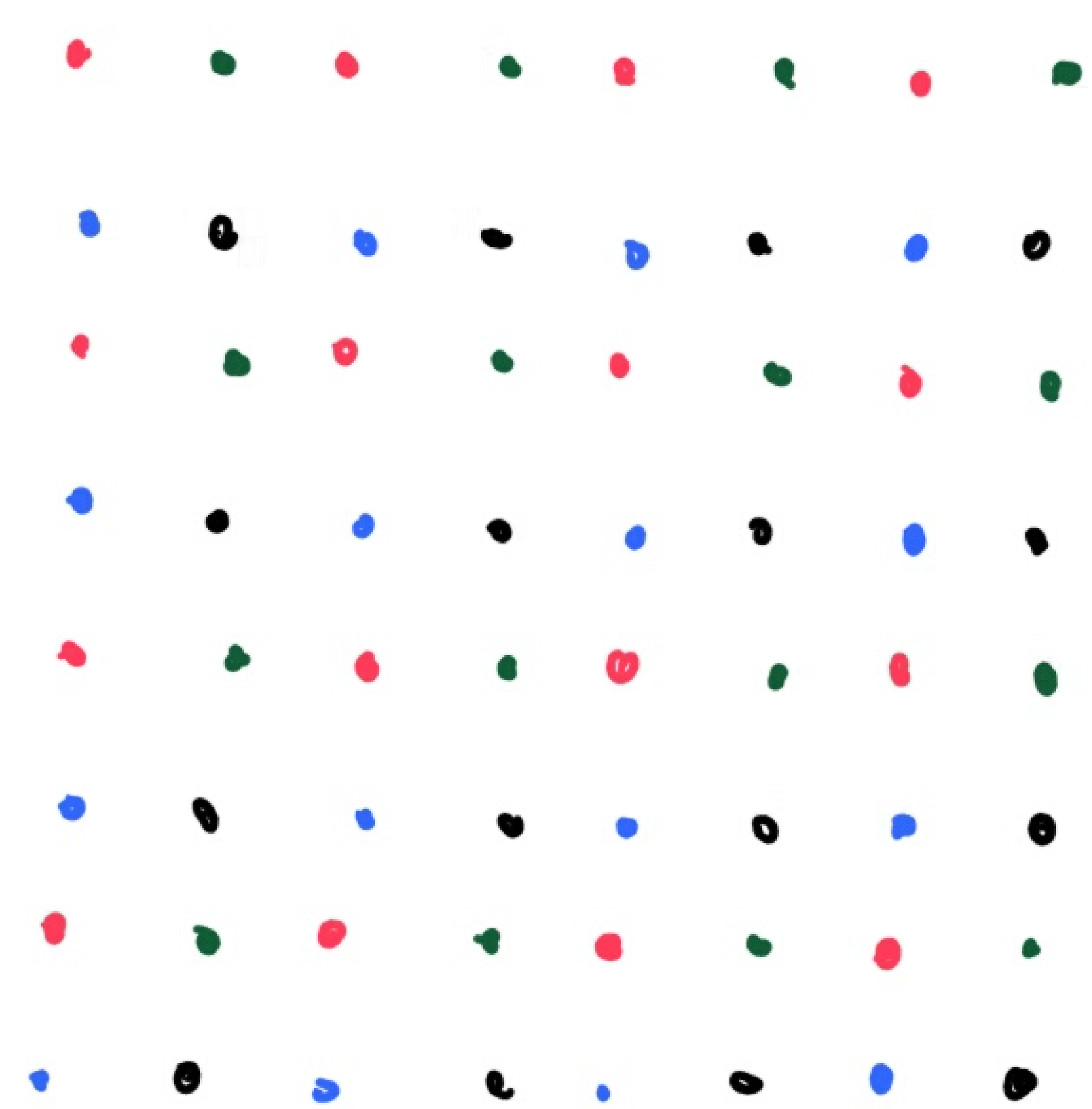
$$\mathcal{H} = M_{4N}(\mathbb{C}) \otimes \mathbb{C}^4$$ has a Dirac operator

(JWB + James Gaunt)

$$\mathcal{H}_x \cong \left(M_N(\mathbb{C}) \oplus M_N(\mathbb{C}) \oplus M_N(\mathbb{C}) \oplus M_N(\mathbb{C}) \right) \otimes \mathbb{C}^4$$

$N = 4$

picture of momentum lattice



eigenvalues of D^2

$$[k]^2 + [l]^2$$

$$[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$$