UCL DEPARTMENT OF MATHEMATICS

Hadamard states for quantum Abelian duality

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M. Benini, M. C., C. Dappiaggi, *Annales Henri Poincaré* **18** (2017), 3325–3370.

Abelian duality: what's it all about

Vacuum Maxwell equations

$$\operatorname{div} E = 0, \qquad \operatorname{div} B = 0,$$
$$\operatorname{curl} E + \frac{\partial B}{\partial t} = 0, \qquad \operatorname{curl} B - \frac{\partial E}{\partial t} = 0.$$

The equations are symmetric under duality:

$$(E, B) \longleftrightarrow (B, -E).$$

A more geometric perspective

(M, g) 4-dimensional g.h. space-time.
$F \in \Omega^2(M)$ electromagnetic tensor.

Maxwell equations become

$$\mathrm{d}\, F = 0, \qquad \delta\, F = 0.$$

Duality:

$$F \longleftrightarrow *F.$$

Fluxes

Let Σ be a Cauchy surface, Ω a 2–dimensional, embedded, closed submanifold of Σ.

Magnetic

$$\int_\Omega F$$

$$[F] \in H^2(M;\mathbb{R})$$

Electric

$$\int_\Omega *F$$

$$[*F] \in H^2(M;\mathbb{R})$$

Yang-Mills approach

Replace F with a $U(1)$-principal bundle with a connection (P, A).

Then the dynamics is governed by the curvature F_A of the connection A:

$$\mathrm{d}\, F_A = 0, \qquad \mathrm{d} * F_A = 0.$$

We gain:

- The theory accounts for the Aharonov-Bohm effect
- The magnetic flux is discretised: the characteristic class of P is in $H^2(M;\mathbb{Z})$

Issues:

- The electro-magnetic duality is broken
- The electric flux is not discretised, $[*F_A] \in H^2(M;\mathbb{R})$

Question: is it possible to devise a theory naturally implementing the Abelian duality, accounting for the Aharonov-Bohm effect and the discretisation of electric and magnetic fluxes?

The idea

The whole problem is originated by $*F_A$ not being the curvature of a connection.

Consider two copies of our principal bundle, (P, A) and $(\tilde{P}, \tilde{A})$, with the constraint

$$F_A = *F_{\tilde{A}}. \tag{1}$$

Properties:

- Duality: $(P, A) \longleftrightarrow (\tilde{P}, \tilde{A})$;
- Dynamics: $\mathrm{d}\, F_A = 0 \ \wedge \ \mathrm{d}\, F_{\tilde{A}} = 0 \quad \Rightarrow \quad \mathrm{d} * F_A = 0 \ \wedge \ \mathrm{d} * F_{\tilde{A}} = 0$;
- Fluxes: characteristic classes $c(P), c(\tilde{P}) \in H^2(M; \mathbb{Z})$.

Becker, Benini, Schenkel and Szabo (2017)

$$\begin{cases} \text{Locally covariant QFT} \\ \text{Differential cohomology} \end{cases} \quad \Rightarrow \quad \text{Abelian duality}$$

C. Becker, M. Benini, A. Schenkel, R. J. Szabo, *Abelian duality on globally hyperblic spacetimes*, Communications in Mathematical Physics **349** (2017), 361–392.

Differential cohomology

Differential cohomology is a (contraviariant) functor $\widehat{H}^*(\cdot;\mathbb{Z}) : \mathsf{Man} \to \mathsf{Ab}$ together with four natural transformations s.t.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{H^{k-1}(M;\mathbb{R})}{H^{k-1}_{\text{free}}(M;\mathbb{Z})} & \longrightarrow & \dfrac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathbb{Z}}(M)} & \longrightarrow & d\Omega^{k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & H^{k-1}(M;\mathbb{T}) & \xrightarrow{\kappa} & \hat{H}^k(M;\mathbb{Z}) & \xrightarrow{\text{curv}} & \Omega^k_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \text{char}} & & \downarrow & & \\
0 & \longrightarrow & H^k_{\text{tor}}(M;\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{Z}) & \longrightarrow & H^k_{\text{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Cheeger-Simons differential characters

A model for differential cohomology are

Differential Characters

A k-differential character $h \in \widehat{H}^k(M;\mathbb{Z})$ is a homomorphism $h : Z_{k-1}(M) \to \mathbb{T}$ s.t. there exists $\omega \in \Omega^k(M)$ for which

$$h(\partial\gamma) = \int_\gamma \omega \mod \mathbb{Z} \qquad \forall \gamma \in C_k(M).$$

Hence: $F_A \quad \longrightarrow \quad \omega = \operatorname{curv} h.$

- Freed, Moore, Segal '07
  Becker, Schenkel, Szabo '14
  Becker, Benini, Schenkel, Szabo '17

Configuration space

$$\mathfrak{C}^k(M) = \{(h, \tilde{h}) \in \widehat{H}^k(M;\mathbb{Z}) \times \widehat{H}^{m-k}(M;\mathbb{Z}) : \operatorname{curv} h = * \operatorname{curv} \tilde{h}\}.$$

Each pair $(h, \tilde{h})$ comes with corresponding pairs of curvatures and Chern classes.

Henceforth, assume Σ compact.

$$
\begin{array}{ccccc}
\mathsf{Top}^k_{f,1}(M) & \rightarrowtail & G_1 & \twoheadrightarrow & \mathsf{Dyn}^k(M) \\
\downarrow & & \downarrow & & \downarrow \\
G_4 & \rightarrowtail & \mathfrak{C}^k(M) & \twoheadrightarrow & G_2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Top}^k_t(M) & \rightarrowtail & G_3 & \twoheadrightarrow & \mathsf{Top}^k_{f,2}(M)
\end{array}
$$

Symplectic structure:

$$\sigma : \mathfrak{C}^k(M) \times \mathfrak{C}^k(M) \to \mathbb{T}, \qquad \sigma((h,\tilde{h}),(h',\tilde{h}')) = \int_\Sigma \tilde{h} \cdot h' - \tilde{h}' \cdot h$$

Observables:

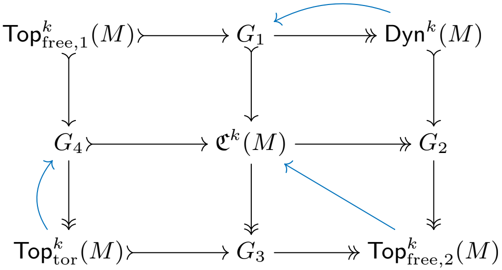
$$\sigma(\,\cdot\,,(h,\tilde{h})) : \mathfrak{C}^k(M) \to \mathbb{T}$$

Duality:

$$\zeta : \mathfrak{C}^k(M) \to \mathfrak{C}^{m-k}(M), \qquad \zeta(h,\tilde{h}) = (\tilde{h}, (-1)^{k(m-k)+1} h)$$

Constructing states

Henceforth, assume Σ compact, $g = -dt \otimes dt + h_\Sigma$ ultrastatic.



Symplectic decomposition:

$$(\mathfrak{C}^k(M), \sigma) = (\mathsf{Dyn}^k(M), \sigma_{\mathrm{dyn}}) \oplus (\mathsf{Top}^k_{\mathrm{free}}(M), \sigma_{\mathrm{free}}) \oplus (\mathsf{Top}^k_{\mathrm{tor}}(M), \sigma_{\mathrm{tor}})$$

Properties

- Compatible with duality
- $\mathsf{Dyn}^k(M)$ dynamical sector, governed by PDE
- $\mathsf{Top}^k_{\mathrm{free}}(M)$, $\mathsf{Top}^k_{\mathrm{tor}}(M)$ topological sectors, finitely many degrees of freedom

Constructing states (2)

The symplectic decomposition entails a factorisation at the level of the C^*-algebra:

$$\mathfrak{CCR}(\mathfrak{C}^k(M), \sigma) \cong \mathcal{W}(\mathsf{Dyn}^k(M)) \otimes \mathcal{W}(\mathsf{Top}^k_{\mathrm{free}}(M)) \otimes \mathcal{W}(\mathsf{Top}^k_{\mathrm{tor}}(M))$$

This enables us to construct states separately for each factor

$$\omega = \omega_{\mathrm{dyn}} \otimes \omega_{\mathrm{free}} \otimes \omega_{\mathrm{tor}}$$

Requirements

- For the dynamical sector, we seek a *Hadamard state*
- Several choices for the state on the topological sectors
- States should be invariant under duality

$\Rightarrow$ duality unitarily implemented in the GNS Hilbert space!

Dynamical sector

The dynamical sector is given by $\mathsf{Dyn}^k(M) = \mathrm{d}\,\Omega^{k-1}(M) \cap * \mathrm{d}\,\Omega^{m-k-1}(M)$.
With $-\Delta_\Sigma\, u_j = \lambda_j^2\, u_j$,

PDE

$$\mathrm{d}\,A = * \mathrm{d}\,\tilde{A}$$

Initial conditions

$$\mathrm{d}_\Sigma\, A_0 = \sum_i \alpha_i\, \mathrm{d}_\Sigma\, u_i$$

$$\mathrm{d}_\Sigma\, \tilde{A}_0 = \sum_i \tilde{\alpha}_i * u_i$$

Solution

$$A = \sum_i \left[\alpha_i \cos(\lambda_i t) + (-1)^{k(m-k)} \tilde{\alpha}_i \lambda_i^{-1} \sin(\lambda_i t)\right] u_i$$

$$\tilde{A} = \sum_i \left[(-1)^{km+1} \alpha_i \lambda_i^{-i} \sin(\lambda_i t) + (-1)^k \tilde{\alpha}_i \lambda_i^{-2} \cos(\lambda_i t)\right] *_\Sigma d_\Sigma u_i$$

Dynamical state

$$\omega_{\mathrm{dyn}} : \mathcal{W}(\mathsf{Dyn}^k(M)) \to \mathbb{C}$$

$$\omega_{\mathrm{dyn}}(\mathrm{d}\,A) = e^{-\frac{1}{4}\sum_i \frac{1}{\lambda_i}(\lambda_i^2 \alpha_i^2 + \tilde{\alpha_i}^2)}$$

Properties

- Ground state
- Hadamard state
- Invariant under duality & s.t. symmetries

An example: 2D case

$M = \mathbb{R} \times \mathbb{S}^1$, $g = -dt \otimes dt + d\theta \otimes d\theta$

$$\mathfrak{C}^1(M) = \mathrm{d}\, C^\infty \cap * \mathrm{d}\, C^\infty(M) \oplus \mathbb{T}^2 \oplus \mathbb{Z}^2$$

$$h(t,\theta) = h_0 + n\theta + \Bigg(-\tilde{n}t + \sum_{k=1}^{\infty} \{ -b_k^- \cos[k(t-\theta)] - b_k^+ \cos[k(t+\theta)] + a_k^- \sin[k(t-\theta)] + a_k^+ \sin[k(t+\theta)] \} \mod \mathbb{Z} \Bigg)$$

$$\tilde{h}(t,\theta) = \tilde{h}_0 + \tilde{n}\theta + \Bigg(-nt + \sum_{k=1}^{\infty} \{ -b_k^- \cos[k(t-\theta)] + b_k^+ \cos[k(t+\theta)] + a_k^- \sin[k(t-\theta)] - a_k^+ \sin[k(t+\theta)] \} \mod \mathbb{Z} \Bigg)$$

No zero modes in the dynamical sector! [Cfr. Schubert, 2013]

State

$$\omega_{\mathrm{dyn}}(\mathcal{W}(d\varphi)) = \exp\left(-\frac{1}{4} \sum_{k=1}^{\infty} k \left\{ (a_k^+)^2 + (b_k^+)^2 + (a_k^-)^2 + (b_k^-)^2 \right\} \right)$$

Topological sector

$$\mathsf{Top}^k_{\text{free}}(\mathbb{R}\times\mathbb{S}^2)\simeq H^1(M;\mathbb{T})^2\oplus H^1(M;\mathbb{Z})^2\simeq\mathbb{T}^2\oplus\mathbb{Z}^2$$

$$\omega_{\text{free}}:\mathcal{W}(\mathsf{Top}^k_{\text{free}}(M))\to\mathbb{C}$$

$$\mathcal{W}(u,\tilde{u},v,\tilde{v})\mapsto\begin{cases}1 & \text{if } v=\tilde{v}=0\\ 0 & \text{otherwise}\end{cases}$$

- Basis for the GNS Hilbert space labelled by $|v,\tilde{v}\rangle$
- **Interpretation**: two particles on a circle, with initial positions $(u,\tilde{u})$ and initial momenta $(v,\tilde{v})$
- Unitary duality operator: $U|v,\tilde{v}\rangle=|\tilde{v},v\rangle$

Translation operators

$$\mathcal{T}(v'):=\pi_{\omega_{\text{free}}}(\mathbb{W}(0,0,v',0))$$
$$\mathcal{T}(v')|v,\tilde{v}\rangle=|v+v',\tilde{v}\rangle$$
$$\tilde{\mathcal{T}}(\tilde{v}'):=\pi_{\omega_{\text{free}}}(\mathbb{W}(0,0,0,\tilde{v}'))$$
$$\tilde{\mathcal{T}}(\tilde{v}')|v,\tilde{v}\rangle=|v,\tilde{v}+\tilde{v}'\rangle$$

Momentum operators

$$\Pi|v,\tilde{v}\rangle=v|v,\tilde{v}\rangle$$
$$\tilde{\Pi}|v,\tilde{v}\rangle=\tilde{v}|v,\tilde{v}\rangle$$

Summary

- Differential cohomology is an effective tool to describe Abelian gauge theory with duality: topological QM information + dynamics

- At the level of the observables, we can perform an orthogonal symplectic decomposition into dynamical and topological sectors

$$\mathfrak{C}^k(M) = \mathsf{Dyn}^k(M) \oplus \mathsf{Top}^k_{\mathrm{free}}(M) \oplus \mathsf{Top}^k_{\mathrm{tor}}(M)$$

- Factorisation at the level of C^*-algebra allows to construct states on each sector separately

$$\omega = \omega_{\mathrm{dyn}} \otimes \omega_{\mathrm{free}} \otimes \omega_{\mathrm{tor}}$$

- Duality naturally encoded in the model and implemented in the GNS triple as unitary operator

- Ground Hadamard state also in $1+1$ dimensions, due to absence of 0-modes

References

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  M.Sc. thesis, University of Pavia (2016), arXiv:1611.09055 [math-ph].
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THANK YOU FOR YOUR ATTENTION!