Aspects of defects and integrability

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Contents

- Boundaries and defects (eg impurities, shocks, dislocations) are ubiquitous in nature
- What are their properties within an integrable field theory in two-dimensional space-time?
- How are boundaries and defects related?
  - Examples of integrable defects and the special role played by energy-momentum conservation and Bäcklund transformations
  - Solitons scattering with defects and some curious effects
  - Defects in integrable quantum field theory and transmission matrices
  - Boundaries revisited
- With - P Bowcock, C Robertson (Durham-Maths)
C Zambon (Durham-Physics)
D Hills, R Parini (York)

An integrable discontinuity - Bowcock, EC, Zambon (2003)

Start with a single selected point on the x-axis, say x_0, and denote the field to the left ($x < x_0$) by u, and to the right ($x > x_0$) by v:

$$\cdots \qquad u(x,t) \qquad x_0 \qquad v(x,t) \qquad \cdots$$

Field equations in separated domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < x_0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > x_0, \quad \partial^2 = \partial_t^2 - \partial_x^2$$

- How can the fields u, v be 'sewn' together at x_0?
- If the wave equations are nonlinear but 'integrable' are there sewing conditions that preserve the integrability?
  - Not so easy, see: Goodman, Holmes, Weinstein (2002)
  - sine-Gordon, KdV, nonlinear Schrödinger ...

• A simple example (δ-impurity) would be to put

$$u(x_0,t) = v(x_0,t), \quad u_x(x_0,t) - v_x(x_0,t) = 2\lambda u(x_0,t),$$

with linear wave equations for u and v.

• Typically, there is reflection and transmission:

$$u = e^{-i\omega t}\left(e^{ikx} + R\,e^{-ikx}\right), \quad v = e^{-i\omega t}\,T\,e^{ikx}, \quad \omega^2 = k^2$$

with

$$R = -\frac{\lambda e^{2ikx_0}}{ik+\lambda}, \quad T = \frac{ik}{ik+\lambda}$$

• There is a distinguished point, translation symmetry is lost and conservation laws - at least some of them (for example, energy-momentum) - will be violated unless the impurity contributes compensating terms.

Consider the field contributions to energy-momentum:

$$P^\mu = \int_{-\infty}^{x_0} dx\, T^{0\mu}(u) + \int_{x_0}^{\infty} dx\, T^{0\mu}(v), \quad \partial_\nu T^{\nu\mu} = 0$$

where the components of $T^{\nu\mu}(u)$ are (similarly with v)

$$T^{00} = \frac{1}{2}\left(u_t^2 + u_x^2\right) + U,\; T^{01} = T^{10} = -u_t u_x,\; T^{11} = \frac{1}{2}\left(u_t^2 + u_x^2\right) - U$$

Using the field equations, can we arrange

$$\frac{dP^\mu}{dt} = -\left[T^{1\mu}(u)\right]_{x=x_0} + \left[T^{1\mu}(v)\right]_{x=x_0} = -\frac{dD^\mu(u,v)}{dt}$$

with the right hand side depending only on the fields at $x = x_0$?

If so, $P^\mu + D^\mu$ is conserved with D^μ being the defect contribution.

- It turns out that only a few possible sewing conditions (and bulk potentials $U,\; V$) are permitted for this to work.

Consider the field contribution to energy and calculate

$$\frac{dP^0}{dt} = [u_x u_t]_{x_0} - [v_x v_t]_{x_0}.$$

Choosing sewing conditions of the form

$$u_x = v_t + X(u,v), \quad v_x = u_t + Y(u,v), \quad \text{at } \; x = x_0$$

we find

$$\frac{dP^0}{dt} = u_t X - v_t Y.$$

This is a total time derivative if

$$X = -\frac{\partial D^0}{\partial u}, \quad Y = \frac{\partial D^0}{\partial v},$$

for some D^0. Then

$$\frac{dP^0}{dt} = -\frac{dD^0}{dt}.$$

- Expected anyway since time translation remains good.

On the other hand, for momentum

$$\frac{dP^1}{dt} = -\left[\frac{u_t^2+u_x^2}{2} - U(u)\right]_{x_0} + \left[\frac{v_t^2+v_x^2}{2} - V(v)\right]_{x_0}$$

$$= \left[-v_tX + u_tY - \frac{X^2-Y^2}{2} + U - V\right]_{x_0} = -\frac{dD^1(u,v)}{dt}$$

This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus

$$X = -\frac{\partial D^0}{\partial u} = \frac{\partial D^1}{\partial v}, \quad Y = \frac{\partial D^0}{\partial v} = -\frac{\partial D^1}{\partial u},$$

In other words the fields at the defect should satisfy:

$$\frac{\partial^2 D^0}{\partial v^2} = \frac{\partial^2 D^0}{\partial u^2}, \quad \frac{1}{2}\left(\frac{\partial D^0}{\partial u}\right)^2 - \frac{1}{2}\left(\frac{\partial D^0}{\partial v}\right)^2 = U(u) - V(v).$$

Highly constraining - just a few possible combinations U, V, D^0 ...

- sine-Gordon, Liouville, massless free, or, massive free.

For example, if $U(u) = m^2u^2/2,\ V(v) = m^2v^2/2$, D^0 turns out to be

$$D^0(u,v) = \frac{m\sigma}{4}(u+v)^2 + \frac{m}{4\sigma}(u-v)^2,$$

and σ is a free parameter.

- Note: the Tzitzéica (aka BD, MZS, $a_2^{(2)}$ affine Toda) potential

$$U(u) = e^u + 2e^{-u/2}$$

is **not** possible.

- There is a Lagrangian description of this type of defect (type I):

$$\mathcal{L} = \theta(-x+x_0)\mathcal{L}(u)+\delta(x-x_0))\left(\frac{uv_t-u_tv}{2} - D^0(u,v)\right)+\theta(x-x_0))\mathcal{L}(v)$$

In the free case ($m \neq 0$), with a wave incident from the left half-line

$$u = \left(e^{ikx} + Re^{-ikx}\right) e^{-i\omega t}, \;\; v = T\, e^{ikx} e^{-i\omega t}, \;\; \omega^2 = k^2 + m^2,$$

we find:

$$R = 0, \quad T = -\frac{(i\omega - m \sinh\eta)}{(ik + m\cosh\eta)} = -i\,\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}, \;\; \sigma = e^{-\eta}$$

• By design, conserves energy/momentum (no dependence on x_0).

• No bound state (provided η is real).

• By comparison, for δ-impurity:

$$u(x_0, t) = v(x_0, t), \quad u_x(x_0, t) - v_x(x_0, t) = 2\lambda u(x_0, t)),$$

$$R = -\frac{\lambda e^{2ix_0}}{\lambda + ik}, \;\; T = \frac{ik}{\lambda + ik}$$

• Bound state at $k = i\lambda$ if $m > \lambda > 0$. The δ-impurity preserves energy but not momentum (explicit dependence on x_0 in R)

sine-Gordon - Bowcock, EC, Zambon (2003, 2004, 2005)

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), the allowed possibilities are:

$$D^0(u,v) = -2\left(\sigma\cos\frac{u+v}{2} + \sigma^{-1}\cos\frac{u-v}{2}\right),$$

where σ is a constant, to find (putting $x_0 = 0$)

$$\begin{aligned}
x<0: \quad \partial^2 u &= -\sin u,\\
x>0: \quad \partial^2 v &= -\sin v,\\
x=0: \quad u_x &= v_t - \sigma\sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2},\\
x=0: \quad v_x &= u_t + \sigma\sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2}.
\end{aligned}$$

- The final two are a Bäcklund transformation frozen at $x = 0$.
- The defect could be anywhere - essentially topological
- Other features (eg higher spin charges) have been checked.

Solitons and defects - Bowcock, EC, Zambon (2005)

The sine-Gordon model has solitons and antisolitons.

Consider a soliton incident from $x < 0$.

It will not be possible to satisfy the sewing conditions (in general, for all times) unless a similar soliton emerges into the region $x > 0$:

$$x < 0: \quad e^{iu/2} = \frac{1 + iE}{1 - iE},$$
$$x > 0: \quad e^{iv/2} = \frac{1 + izE}{1 - izE},$$
$$E = e^{ax+bt+c}, \qquad a = \cosh\theta, \quad b = -\sinh\theta,$$

where z is to be determined. It is also useful to set $\sigma = e^{-\eta}$.

- We find....

$$z = \text{coth}\left(\frac{\eta - \theta}{2}\right) \qquad \theta > 0$$

Remarks:

- $\eta < \theta$ implies $z < 0$; ie the soliton emerges as an anti-soliton.

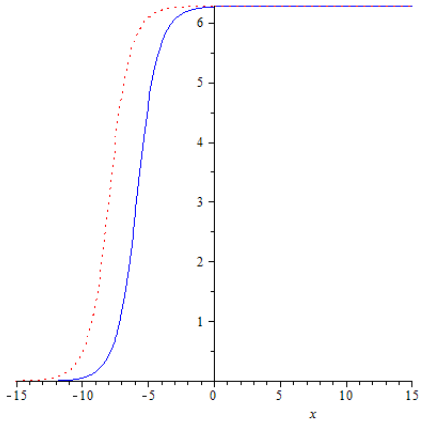
  - the final state will contain a discontinuity of magnitude 4π at $x = 0$.

- $\eta = \theta$ implies $z = \infty$ and there is **no** emerging soliton.

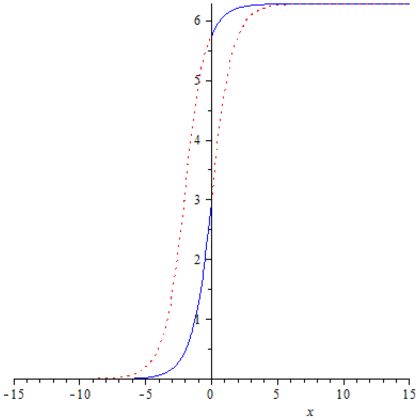
  - the energy-momentum of the soliton is captured by the 'defect'.

  - the topological charge is also captured by a discontinuity 2π.

- $\eta > \theta$ implies $z > 0$; ie the soliton retains its character.



Delayed soliton



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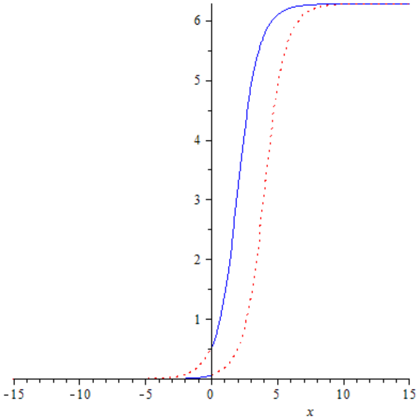

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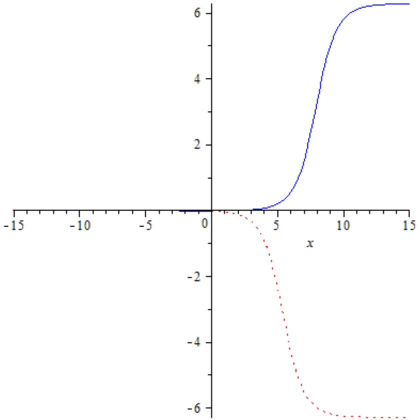
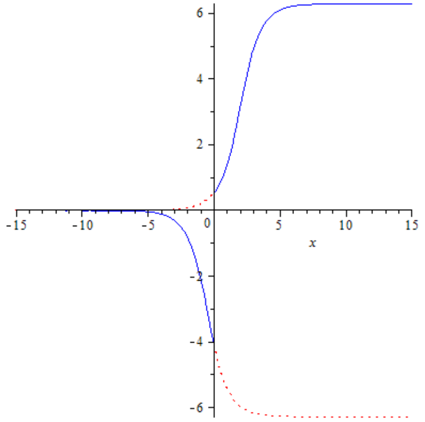
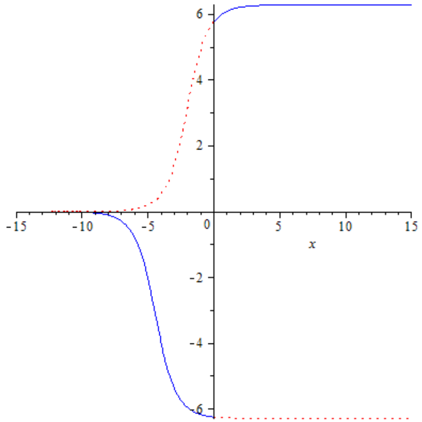


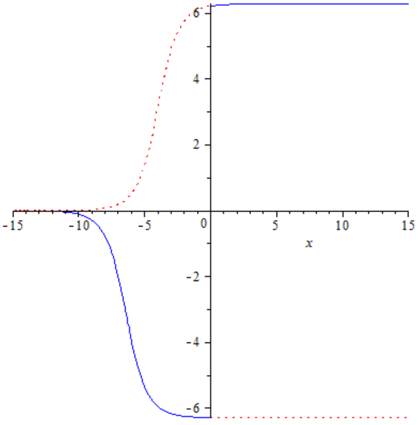
Swapped soliton/anti-soliton



Swapped soliton/anti-soliton



Swapped soliton/anti-soliton



Swapped soliton/anti-soliton

Comments

- Defects at $x = x_1 < x_2 < x_3 < \cdots < x_n$ behave independently

  - each contributes a factor z_i for a total $z = z_1 z_2 \ldots z_n$.

- Each component of a multisoliton is affected separately

  - thus at most one can be 'filtered out'.

- Since a soliton can be absorbed, can a starting configuration with $u = 0$, $v = 2\pi$ decay into a soliton?

  - needs quantum mechanics to provide the probability for decay.

- Contrast Estabrook - Wahlquist (1973)

  - a Bäcklund transformation 'creates' a soliton.

- Defects can also move (with constant speed), and scatter.

Generalisations

- What about Tzitzéica ($a_2^{(2)}$ affine Toda)?
- Multi-component fields - what about other affine Toda field theories?

  - only the $a_n^{(1)}$ affine Toda theories can work - EC, Zambon (2009)

  - Bäcklund transformations are similar - Fordy, Gibbons (1980)
- What about nonlinear Schrödinger, KdV, mKdV, etc, etc?

  - yes, see Caudrelier, Mintchev, Ragoucy (2004,) EC, Zambon (2005), Caudrelier (2008), ...
- Is the setup genuinely integrable? For an alternative (algebraic) approach see Avan, Doikou (2012, 2013); Doikou (2014, 2016)
- What about SUSY? See, for example, Gomes, Ymai, Zimerman (2008); Aguirre, Gomes, Spano, Zimerman (2015)
- How do the sewing conditions work for 'finite gap' solutions of sine-Gordon (or KdV, NLS, etc)? EC, Parini (2016)

Classical type II defect - EC, Zambon (2009)

Consider two relativistic field theories with fields u and v, and add a new degree of freedom $\lambda(t)$ at the defect location ($x = 0$):

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left((u-v)\lambda_t - D^0(\lambda, u, v)\right)$$

Then the usual Euler-Lagrange equations lead to

- equations of motion:

$$\partial^2 u = -\frac{\partial U}{\partial u} \quad x < 0, \qquad \partial^2 v = -\frac{\partial V}{\partial v} \quad x > 0$$

- defect conditions at $x = 0$

$$u_x = \lambda_t - D^0_u \qquad v_x = \lambda_t + D^0_v \qquad (u-v)_t = -D^0_\lambda.$$

As before, consider momentum

$$P^1 = -\int_{-\infty}^{0} dx\, u_t u_x - \int_{0}^{\infty} dx\, v_t v_x ,$$

and seek a functional $D^1(u, v, \lambda)$ such that $P^1_t \equiv -D^1_t$.

As before, implies constraints on $U,\ V,\ D^1$.
Putting $q = (u - v)/2,\ p = (u + v)/2$ these are:

$$D^0_p = -D^1_\lambda \qquad D^0_\lambda = -D^1_p$$

implying

$$D^0 = f(p + \lambda, q) + g(p - \lambda, q) \qquad D^1 = f(p + \lambda, q) - g(p - \lambda, q)$$

and

$$\frac{1}{2}(D^0_\lambda D^1_q - D^0_q D^1_\lambda) = U(u) - V(v)$$

- Powerful constraint on $f,\ g$ since λ does not enter the right side - what is the general solution?

Note:

- Now possible to choose f, g for potentials U, V any one of sine-Gordon, Liouville, Tzitzéica, or free massive or massless.
- Tzitzéica:

$$U(u) = (e^u + 2\,e^{-u/2} - 3), \quad V(v) = (e^v + 2\,e^{-v/2} - 3)$$

and the defect potential $D^0(\lambda, p, q)$ is given by

$$\begin{aligned} D^0 = \quad & 2\sigma \left(e^{(p+\lambda)/2} + e^{-(p+\lambda)/4} \left(e^{q/2} + e^{-q/2} \right) \right) \\ & + \frac{1}{\sigma} \left(8\,e^{-(p-\lambda)/4} + e^{(p-\lambda)/2} \left(e^{q/2} + e^{-q/2} \right)^2 \right) \end{aligned}$$

- In sine-Gordon the type-II defect is new with two free parameters - in a sense it is two 'fused' type-I defects - EC, Zambon (2010)
- Other field theories? - yes, *but not yet all*, Robertson (2014); Bowcock, Umpleby (2008); Bowcock, Bristow (2016)

Defects in quantum field theory

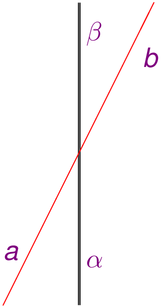
- Expect Soliton-defect scattering - pure transmission compatible with the bulk S-matrix
- Expect Topological charge will be preserved but may be exchanged with the defect
- Expect For each type of defect there may be several types of transmission matrix (eg in sine-Gordon expect two different transmission matrices since the topological charge on a defect can only change by ± 2 as a soliton/anti-soliton passes).
- Expect Not all transmission matrices need be unitary (eg in sine-Gordon one is a ‘resonance’ of the other)
- Questions Relationship between different types of defect; assemblies of defects, defect-defect scattering; fusing defects; ...

A transmission matrix is intrinsically infinite-dimensional:

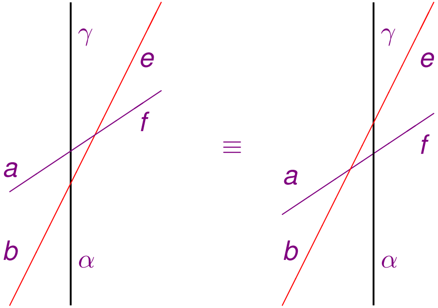
$$T_{a\alpha}^{b\beta}(\theta,\eta), \quad a+\alpha=b+\beta$$

where α, β and a, b are defect and particle (eg soliton) labels respectively (typically they will be sets of weights); and η is a collection of defect parameters.

Schematically:



Schematic compatibility relation - Delfino, Mussardo, Simonetti (1994)



$$S_{ab}^{cd}(\Theta)\, T_{d\alpha}^{f\beta}(\theta_a) T_{c\beta}^{e\gamma}(\theta_b) = T_{b\alpha}^{d\beta}(\theta_b) T_{a\beta}^{c\gamma}(\theta_a) S_{cd}^{ef}(\Theta)$$

With $\Theta = \theta_a - \theta_b$ and sums over the 'internal' indices β, c, d.

- For sine-Gordon a solution was known - Konik, LeClair (1999)

Zamolodchikov's sine-Gordon soliton-soliton S-matrix - reminder

$$S_{ab}^{cd}(\Theta) = \rho(\Theta) \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & B & 0 \\ 0 & B & C & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where

$$A(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2},\ B(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1},\ C(\Theta) = q - \frac{1}{q}$$

$$x_a = e^{\gamma\theta_a},\ a = 1,2,\ \Theta = \theta_1 - \theta_2,\ q = e^{i\pi\gamma},\ \gamma = \frac{8\pi}{\beta^2} - 1,$$

and

$$\begin{aligned} \rho(\Theta) &= \frac{\Gamma(1+z)\Gamma(1-\gamma-z)}{2\pi i} \prod_1^\infty R_k(\Theta) R_k(i\pi - \Theta) \\ R_k(\Theta) &= \frac{\Gamma(2k\gamma + z)\Gamma(1 + 2k\gamma + z)}{\Gamma((2k+1)\gamma + z)\Gamma(1 + (2k+1)\gamma + z)},\ z = i\gamma/\pi. \end{aligned}$$

Useful to define the variable $Q = e^{4\pi^2 i/\beta^2} = \sqrt{-q}$.

- K-L solutions have the form

$$T_{a\alpha}^{b\beta}(\theta) = f(q,x) \begin{pmatrix} Q^{\alpha}\, \delta_{\alpha}^{\beta} & q^{-1/2} e^{\gamma(\theta-\eta)}\, \delta_{\alpha}^{\beta-2} \\ q^{-1/2}\, e^{\gamma(\theta-\eta)}\ \delta_{\alpha}^{\beta+2} & Q^{-\alpha}\, \delta_{\alpha}^{\beta} \end{pmatrix}$$

where $f(q,x)$ is not uniquely determined but, for a unitary transmission matrix, should satisfy

$$\begin{aligned} \bar{f}(q,x) &= f(q,qx) \\ f(q,x)f(q,qx) &= \left(1 + e^{2\gamma(\theta-\eta)}\right)^{-1} \end{aligned}$$

- A 'minimal' solution has the following form

$$f(q,x) = \frac{e^{i\pi(1+\gamma)/4}}{1 + ie^{-2\pi iz}}\, \frac{r(x)}{\bar{r}(x)},$$

where it is convenient to put $z = i\gamma(\theta-\eta)/2\pi$ and

$$r(x) = \prod_{k=0}^{\infty} \frac{\Gamma(k\gamma + 1/4 - z)\Gamma((k+1)\gamma + 3/4 - z)}{\Gamma((k+1/2)\gamma + 1/4 - z)\Gamma((k+1/2)\gamma + 3/4 - z)}$$

$$T_{a\alpha}^{b\beta}(\theta) = f(q,x)\begin{pmatrix} Q^{\alpha}\,\delta_{\alpha}^{\beta} & q^{-1/2}e^{\gamma(\theta-\eta)}\,\delta_{\alpha}^{\beta-2} \\ q^{-1/2}\,e^{\gamma(\theta-\eta)}\;\delta_{\alpha}^{\beta+2} & Q^{-\alpha}\,\delta_{\alpha}^{\beta} \end{pmatrix}$$

Remarks (supposing $\theta > 0$) - Bowcock, EC, Zambon (2005):

Tempting to suppose η (possibly renormalized) is the same parameter as in the type I classical model.

- $\eta < 0$ - the off-diagonal entries dominate;
- $\theta > \eta > 0$ - the off-diagonal entries dominate;
- $\eta > \theta > 0$ - the diagonal entries dominate.

• Similar features to the classical soliton-defect scattering.

• The different behaviour of solitons versus anti-solitons (diagonal terms) is a direct consequence of the defect term proportional to

$$\delta(x - x_0)(uv_t - vu_t)/2$$

• $\theta = \eta$ is not special (neither is $z = -1/4$) but there is a simple pole nearby at $z = 1/4$:

$$\theta = \eta - \frac{i\pi}{2\gamma} \to \eta, \text{ as } \beta \to 0$$

This pole is like a resonance, with complex energy,

$$E = m_s \cosh\theta = m_s(\cosh\eta\cos(\pi/2\gamma) - i\sinh\eta\sin(\pi/2\gamma))$$

and a 'width' proportional to $\sin(\pi/2\gamma)$.

• The Zamolodchikov S-matrix has 'breather' poles corresponding to soliton-anti-soliton bound states at

$$\Theta = i\pi(1 - n/\gamma), \; n = 1, 2, ..., n_{\max};$$

use the bootstrap to define the transmission factors for breathers and find for the lightest breather:

$$T(\theta) = -i\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}$$

Type II transmission matrix for sine-Gordon - EC, Zambon (2010)

There is another, more general, set of solutions to the quadratic relations for the transmission matrix:

$$\rho(\theta)\begin{pmatrix} (a_+ Q^{\alpha} + a_- Q^{-\alpha}\, x^2)\,\delta_{\alpha}^{\beta} & x\,(b_+ Q^{\alpha} + b_- Q^{-\alpha})\,\delta_{\alpha}^{\beta-2} \\ x\,(c_+ Q^{\alpha} + c_- Q^{-\alpha})\,\delta_{\alpha}^{\beta+2} & (d_+ Q^{\alpha}\, x^2 + d_- Q^{-\alpha})\,\delta_{\alpha}^{\beta} \end{pmatrix}$$

where $x = e^{\gamma\theta}$.

The free constants satisfy the two constraints

$$a_{\pm}\, d_{\pm} - b_{\pm}\, c_{\pm} = 0$$

These and $\rho(\theta)$are constrained further by crossing and unitarity.

- For a choice of parameters this descibes a type II defect.
- With $a_- = d_+ = 0$ and $b_+ = c_- = 0$ or $b_- = c_+ = 0$ (after a similarity transformation), reduces to the type I solution.
- For other choices of parameters reduces to a direct sum of the Zamolodchikov S-matrix and two infinite dimensional pieces.

Alternative formulation - Weston (2010)

Summary: for Type II

$$T = \rho(x)\begin{pmatrix} xa_+Q^{-N} + x^{-1}a_-Q^N & a \\ a^* & xd_+Q^N + x^{-1}d_-Q^{-N} \end{pmatrix},$$

where a^* and a are 'generalised' raising and lowering operators, respectively,

$$a^*|k\rangle = |k+2\rangle \quad a|k\rangle = F(k)|k-2\rangle \quad N|k\rangle = k|k\rangle,\ k\in\mathbb{Z}$$

$$F(N) = f_0 + f_+Q^{2N} + f_-Q^{-2N}, \quad f_+ = Q^{-2}a_-d_+ \quad f_- = Q^2a_+d_-$$

- T intertwines the coproducts of finite (soliton) and infinite (defect) representations of the Borel subalgebra of $U_q(a_1^{(1)})$.

- Idea generalises to all other quantum algebras allowing (in principle) calculations of associated defect matrices. For some examples see EC, Zambon (2010), Boos et al. (2011).

Defect-defect scattering - type I

$$T_1{}^{b\gamma}_{a\alpha}\, T_2{}^{c\delta}_{b\beta}\, U^{\rho\sigma}_{\gamma\delta} = U^{\delta\gamma}_{\alpha\beta}\, T_2{}^{b\rho}_{a\delta}\, T_1{}^{c\sigma}_{b\gamma}.$$

$$T_i \approx \begin{pmatrix} Q^{N_i} & \beta_i\, x\, A_i \\ \beta_i\, x\, A_i^* & Q^{-N_i} \end{pmatrix}, \quad i = 1, 2$$

where

$$x = e^{\gamma\theta},\ q = e^{i\pi\gamma},\ Q^2 = -q; \quad \beta^* = \beta.$$

Data carried by $\beta_i,\ A_i,\ A_i^*,\ i = 1, 2$ with two sets of mutually commuting generalised annihilation and creation operators.

U is independent of x: equating terms in powers of x leads to the following four equations:

$$\left(\beta_2\, Q^{N_1}\, A_2 + \beta_1\, Q^{-N_2}\, A_1\right)\, U = U\, \left(\beta_1\, Q^{N_2}\, A_1 + \beta_2\, Q^{-N_1}\, A_2\right)$$

$$\left(\beta_1\, Q^{N_2}\, A_1^* + \beta_2\, Q^{-N_1}\, A_2^*\right)\, U = U\, \left(\beta_2\, Q^{N_1}\, A_2^* + \beta_1\, Q^{-N_2}\, A_1^*\right)$$

$$Q^{N_1+N_2}\, U = U\, Q^{N_1+N_2},\ \ A_1\, U\, A_1 = A_2\, U\, A_2$$

$$U=\sum_{k=-\infty}^{\infty} A_1^k A_2^{-k} U_k(N_1,N_2,\lambda), \quad \lambda=\beta_1/\beta_2$$

Then

$$U_{k+2}(N_1,N_2,\lambda)=U_k(N_1-2,N_2+2,\lambda)$$

$$U_{2l}(N_1,N_2,\lambda)=U_0(N_1-2l,N_2+2l,\lambda)$$

$$U_{2l+1}(N_1,N_2,\lambda)=U_1(N_1-2l,N_2+2l,\lambda).$$

and

$$\left(\lambda Q^{-N_2}A_1+Q^{N_1}A_2\right)U=U\left(Q^{-N_1}A_2+\lambda Q^{N_2}A_1\right),$$

$$Q^{N_1-2}U_1^{(N_1,N_2)}+\lambda Q^{-N_2}U_0^{(N_1,N_2)}=\lambda Q^{N_2}U_0^{(N_1-2,N_2)}+Q^{-N_1}U_1^{(N_1,N_2-2)}$$

$$Q^{N_1}U_0^{(N_1,N_2+2)}+\lambda Q^{-N_2}U_1^{(N_1+2,N_2)}=\lambda Q^{N_2+2}U_1^{(N_1,N_2)}+Q^{-N_1}U_0^{(N_1,N_2)}$$

Formal generating functionals

$$U(x,y)=\sum_{n,m} x^n y^m U_0(n,m), \quad V(x,y)=\sum_{n,m} x^n y^m U_1(n,m)$$

Then

$$\begin{aligned}\lambda U(x,y/Q)+Q^{-2}V(Qx,y) &= \lambda x^2 U(x,Qy)+y^2 V(x/Q,y)\\ (\lambda/x^2)V(x,y/Q)+(1/y^2)U(Qx,y) &= \lambda Q^2 V(x,Qy)+U(x/Q,y).\end{aligned}$$

These can be written slightly more symmetrically by rearranging and putting $r=Q,\ s=\lambda Q$:

$$\begin{aligned} x\,U(x,ry)-x^{-1}\,U(x,r^{-1}y) &= \frac{y}{xs}\left((ry)^{-1}\,V(rx,y)-ry\,V(r^{-1}x,y)\right)\\ y^{-1}\,U(rx,y)-y\,U(r^{-1}x,y) &= \frac{ys}{x}\left(rx\,V(x,ry)-(rx)^{-1}\,V(x,r^{-1}y)\right).\end{aligned}$$

- What is the general solution?

Further questions....

- Some alternative views and other aspects are discussed in several places. Eg - Habibullin, Kundu (2008); Bajnok, Simon (2008)

- Other Toda models - defects can be constructed for some other affine Toda models, eg the $a_r^{(1)},\ (c_n^{(1)},\ d_{n+1}^{(2)}),\ a_{2n}^{(2)}, d_n^{(1)}$

Bowcock, EC, Zambon (2004), EC, Zambon (2007, 2010, 2011), Robertson (2014), Bowcock, Bristow (2016)

What about all the others?

- Once the question is answered....

- ....What are the general integrable boundary conditions for all affine Toda field theories?

Thank you!