Fermionic Projectors and Hadamard States

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''To the memory of Rudolf Haag, one of the founding fathers of the algebraic quantum field theory''

Outline

- **The algebraic approach to QFT**

- **Quasi-free states for CAR algebras**

- **Fermionic Projector and the Hadamard condition:**
  - **FP states in a strip of spacetime**
  - **FP states and the mass oscillation property**
  - **FP states via the Møller operator**

- **Conclusions**

Based on:

▶ F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*

▶ N. Drago, S. M.: *"A new class of Fermionic Projectors: Møller operators and mass oscillation properties."* (work in progress)

Algebraic Quantum Field Theory

AQFT - I: Dirac field

- **Spinor bundle** $S\mathcal{M} \simeq \mathcal{M} \times \mathbb{C}^4$ and **cospinor bundle** $S^*\mathcal{M} \simeq \mathcal{M} \times (\mathbb{C}^4)^*$, with $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ 4-dim *globally hyperbolic spacetime* :

$$ds^2 = \beta^2 dt^2 - h_t; \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in \text{Riem}(\Sigma); \forall t \in \mathbb{R}.$$

- Spinors $\psi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$ and cospinors $\varphi \in C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- Dirac conjugation map: $A : C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \underset{(\leftarrow)}{\longrightarrow} C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- **Spin scalar product**: $\prec \cdot | \cdot \succ_x : C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$\prec \psi | \widetilde{\psi} \succ_x := \big((A\psi) \widetilde{\psi} \big)(x).$$

- **spacetime inner product**: $<\cdot|\cdot> : C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_c(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$<\psi|\vartheta> = \int_\mathcal{M} \prec \psi|\vartheta \succ_x d\mu_g .$$

AQFT - II: Dynamics

- **Dirac operator** on SM and its dual on S^*M:

$$\mathcal{D}\psi_m \doteq (i\gamma^\mu \nabla_\mu + B - m)\psi_m = 0, \qquad \mathcal{D}^*\varphi_m = (-i\gamma^\mu \nabla_\mu + B - m)\varphi_m = 0.$$

- **Causal propagators:** $G^{(*)} : C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)}) \to C_{sc}^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$

$$\mathcal{D}^{(*)} \circ G^{(*)} = 0 = G^{(*)} \circ \mathcal{D}^{(*)}|_{C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})}$$

$$supp(G^{(*)}(f)) \subseteq J(supp(f)), \qquad \forall f \in C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$$

- **Hilbert spaces:**

$$\mathcal{H}_m^s := \overline{\left(\mathcal{S}ol(\mathcal{D}) \simeq \frac{C_c^\infty(\mathcal{M}, \mathbb{C}^4)}{\mathcal{D}C_c^\infty(\mathcal{M}, \mathbb{C}^4)} , \; (\cdot \mid \cdot)_m^s \doteq \int_\Sigma \prec \cdot \mid \psi \cdot \succ_x d\Sigma \right)}$$

$$\mathcal{H}_m^c := \overline{\left(\mathcal{S}ol(\mathcal{D}^*)) \simeq \frac{C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^*)}{\mathcal{D}^* C_c^\infty(\mathcal{M}, \mathbb{C}^4)^*)} , \; (\cdot \mid \cdot)_m^c \doteq \int_\Sigma \prec A^{-1} \cdot \mid \psi A^{-1} \cdot \succ_x d\Sigma \right)}$$

AQFT - III: CAR Algebra

- unital Borchers-Uhlmann $*$-algebra: $\quad \mathcal{A} = \bigoplus_{k=0}^{\infty} \left(\mathcal{S}ol(\mathcal{D}_m) \oplus \mathcal{S}ol(\mathcal{D}_m^*) \right)^{\otimes k}$

  - $\mathbb{1} = \{1, 0, 0, \ldots\}, \Phi(\psi_m) = \left\{ 0, \binom{\psi_m}{0}, 0, \ldots \right\}, \Psi(\varphi_m) = \left\{ 0, \binom{0}{\varphi_m}, 0, \ldots \right\}$

  - $*$-operation: $\left\{ 0, 0, \binom{\psi_m}{\varphi_m} \otimes \binom{\widetilde{\psi}_m}{\widetilde{\varphi}_m}, \ldots \right\}^* = \left\{ 0, 0, \binom{A^{-1}\widetilde{\varphi}_m}{A\widetilde{\psi}_m} \otimes \binom{A^{-1}\varphi_m}{A\psi_m}, \ldots \right\}$

- We encode the CARs in the $*$-ideal $\mathcal{I} \subset \mathcal{A}$ generated by:

  - $\Phi(\psi_m) \otimes \Phi(\widetilde{\psi}_m) + \Phi(\widetilde{\psi}_m) \otimes \Phi(\psi_m)$

  - $\Psi(\varphi_m) \otimes \Psi(\widetilde{\varphi}_m) + \Psi(\widetilde{\varphi}_m) \otimes \Psi(\varphi_m)$

  - $\Psi(\varphi_m) \otimes \Phi(\psi_m) + \Phi(\psi_m) \otimes \Psi(\varphi_m) - \left(A^{-1}\varphi_m \mid \psi_m \right)_m^s \mathbb{1}$

- **Algebra of fields:** $\quad \mathcal{F} \doteq \dfrac{\mathcal{A}}{\mathcal{I}}$

AQFT - IV: States

- **Algebraic state** $\omega : \mathcal{F} \to \mathbb{C}$ such that: $\omega(\mathbb{1}) = 1$, $\qquad \omega(h^*h) \geq 0$, $\quad \forall\, h \in \mathcal{F}$.

N.B.: Choosing a state ω is equivalent to assigning $\omega_n(h_1, \ldots, h_n)$ $\forall n \in \mathbb{N}$ and $\forall h_i \in \mathcal{F}$.

- **Quasi-free states:** $\qquad \omega_{2n+1}(h_1, \ldots, h_{2n+1}) = 0$

$$\omega_{2n}(h_1, \ldots, h_{2n}) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big).$$

Question: *Are all states physically acceptable?*

Of course not! Minimal physical requirements are:

  i) covariant construction of Wick polynomials to deal with interactions,

 ii) same UV behaviour of the Minkowski vacuum,

iii) finite quantum fluctuations of all observables.

Answer: **Hadamard States**

AQFT - V: Hadamard States

- A (quasi-free) state ω satisfies the **Hadamard condition** if and only if

$$WF(\omega_2) = \left\{ (x, y, \xi_x, \xi_y) \in T^* M^{\otimes 2} \setminus 0 \mid (x, \xi_x) \sim (y, -\xi_y), \quad \xi_x \rhd 0 \right\}.$$

Question: *How many Hadamard states do we know?*

- deformation arguments

- static spacetime

- pseudodifferential calculus

- holographic techniques

Question: *Does there exist an <u>explicit</u> method that which <u>does not</u> make use of use symmetries?*

Quasi-free states for

CAR algebras

Araki's characterisation

[H. Araki: *"On quasifree states of CAR and Bogoliubov automorphisms"*.]

Lemma 3.2: For any quasi-free state ω on $\mathcal{F}$ there exists $Q \in \mathcal{B}(\mathscr{H}_m^s)$ satisfying

(a) $\omega_2\Big(\Psi(\varphi_m)\Phi(\psi_m)\Big) = \big(A^{-1}\varphi_m \mid Q\psi_m\big)_m^s$

(b) $Q + AQA = \mathbb{1}$,

(c) $0 \leq Q = Q^* \leq \mathbb{1}$.

Lemma 3.3: Let Q be a <u>bounded symmetric operator</u> on $\mathscr{H}_m^s$ with the following properties

(a) $Q + AQA = \mathbb{1}$,

(b) $0 \leq Q = Q^* \leq \mathbb{1}$.

Then there exists a <u>unique quasi-free state</u> ω on $\mathcal{F}$ such that

$$\omega_2\Big(\Psi(\varphi_m)\Phi(\psi_m)\Big) = \big(A^{-1}\varphi_m \mid Q\psi_m\big)_m^s$$

- Our motto will be $\implies$ "Split the $\mathscr{H}$ilbert space!"   (...but how?)

The Fermionic Signature Operator

- As before $\mathcal{H}_m^s := \overline{(Sol(\mathcal{D}),\, (\cdot \mid \cdot)_m^s)}$ and let $N(\cdot\,,\cdot) : \mathcal{H}_m^s \times \mathcal{H}_m^s \to \mathbb{C}$ be a <u>symmetric</u> and <u>densely defined</u> sesquilinear form satisfying :

  a) $N(\cdot,\cdot) \leq C(\cdot)\|\cdot\|.$

- The **Fermionic Signature Operator** is $\mathcal{S} \in \mathcal{L}(\mathcal{H}_m^s)$ built out of the Riesz theorem

$$N(\cdot\,,\cdot) = (\cdot \mid \mathcal{S}\,\cdot)_m^s.$$

- Taking $\mathcal{S}^2 := \mathcal{S}^* \mathcal{S}$, we construct the **Fermionic Projector**

$$\chi^{\pm}(S) := \frac{1}{2|S|}(S \pm |S|) : \mathcal{H}_m^s \to \mathcal{H}_m^s.$$

N.B.: If $N(\cdot\,,\cdot) \leq \|\cdot\|\,\|\cdot\|$, then $\mathcal{S} \in \mathcal{B}(\mathcal{H}_m^s)$ is essentially self-adjoint and

$$\chi^{\pm}(S) = \int_{\sigma(S)} \chi(\lambda) dE_\lambda$$

Fermionic Projectors and the Hadamard condition

FP states in a strip of spacetime

- Consider $\Omega \subset (-T, T) \times \Sigma \subset \mathcal{M}$ and $\mathcal{H}_m^s(\Omega) := \overline{\left(\mathcal{S}ol(\mathcal{D}), (\cdot \mid \cdot)_m^s \right)}$.

- The **spacetime inner product**: $<\cdot|\cdot> \; : \; \mathcal{H}_m^s(\Omega) \times \mathcal{H}_m^s(\Omega) \to \mathbb{C}$

$$<\psi_m|\varphi_m> = \int_\Omega \prec\psi_m|\varphi_m\succ_x d\mu_\Omega \qquad \text{(well defined)}$$

$$|<\varphi_m|\psi_m>| \leq c \, \|\varphi_m\|_m \, \|\psi_m\|_m \qquad \text{(bounded)}.$$

- Then the **Fermionic Signature operator** is essentially self-adjoint and

$$\chi^\pm(\mathcal{S}) = \int_\sigma \chi^\pm(\lambda) \, dE_\lambda \quad \xLongrightarrow{Araki} \quad \omega_{FP} \text{ on } \mathcal{F}.$$

Problems:

- For $T \to \infty$ the spacetime inner product is not well defined,

- $\omega_2(x, y)$ is in general not Hadamard!
  [C. Fewster and B. Lang: *"Pure quasifree states of the Dirac field from the fermionic projector"*.]

FP states and the mass oscillation property

The strong mass oscillation property

- Hilbert space: $\mathscr{H}^\infty = \left(\Psi := (\psi_m)_{m \in I = (m_L, m_R)} \ , \ (\cdot \, , \cdot) = \int_I (\cdot | \cdot)_m \, dm \right)$

- Smearing operator: $\mathfrak{p} : \mathscr{H}^\infty \to C^\infty_{\mathrm{sc}}(\mathcal{M}, S\mathcal{M}) \, , \quad \mathfrak{p}\Psi = \int_I \psi_m \, dm$

- Symmetric sesquilinear form: $<\mathfrak{p} \cdot | \mathfrak{p} \cdot> \ : \ \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$

  Q: *Is the new sesquilinear form bounded?*

- The **strong mass oscillation property**:

$$|<\mathfrak{p}\Psi \,|\, \mathfrak{p}\Phi>| \leq c ||\psi_m|| \, ||\varphi_m||$$
$$|<\mathfrak{p}\,T\Psi \,|\, \mathfrak{p}\Phi>| = |<\mathfrak{p}\Psi \,|\, \mathfrak{p}\,T\Phi>|$$

  where $T\Psi = (m\psi_m)_{m \in I}$ and $T\Phi = (m\varphi_m)_{m \in I}$.

- Family of linear operators $(\mathcal{S}_m)_{m \in I}$ with $\mathcal{S}_m \in \mathcal{B}(\mathscr{H}^s_m)$ and $\sup_{m \in I} ||\mathcal{S}_m|| < \infty$

$$m \mapsto (\psi_m \,|\, \mathcal{S}_m \, \varphi_m)_m \text{ is continuous}$$
$$<\mathfrak{p}\Psi | \mathfrak{p}\Phi> = \int_I (\psi_m \,|\, \mathcal{S}_m \, \varphi_m)_m \, dm$$

FP states and the strong mass oscillation property

[F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*.]

$$i\partial_t \psi_m = -\gamma^0 (i\vec{\gamma}\vec{\nabla} + B(t,\vec{x}) - m)\psi_m =: H\psi_m \qquad \text{(Dirac equation)}$$

$$\psi_m|_t = U_m^{t,t_0}\psi_0 + i \int_{t_0}^{t} U_m^{t,\tau} \left(\gamma^0 B \, \psi_m\right)\big|_\tau \, d\tau \qquad \text{(Lippmann-Schwinger equation)}$$

- Setting $B(t,\vec{x}) = 0$: the *strong mass oscillation property* holds and

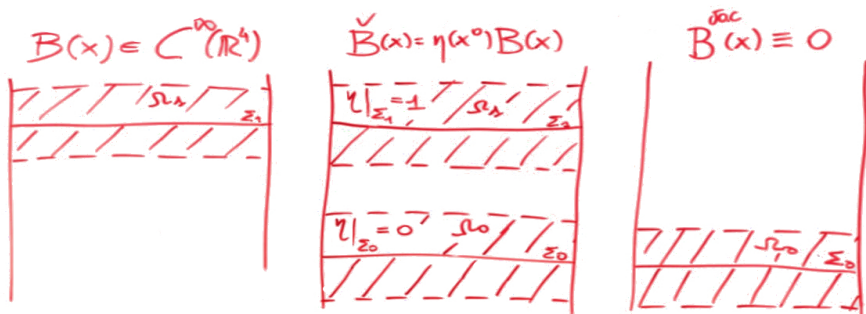$$\mathcal{B}(\mathscr{H}_m^s) \ni \mathcal{S}_m(\vec{\xi}) := \sum_{\xi^0 = \pm\omega(\vec{\xi})} \frac{\slashed{\xi} + m}{2\,\xi^0(\vec{\xi})}\,\gamma^0 \implies \underbrace{\chi^+(\mathcal{S}_m) = \Theta(\xi^0)}_{\text{frequencies splitting}}$$

- Assuming $|B(t)|_{C^2} \leq c \left(1 + |t|^{2+\varepsilon}\right)^{-1}$ and $\int_{-\infty}^{\infty} |B(t)|_{C^0}\, dt < \sqrt{2} - 1$,

$$\mathcal{B}(\tilde{\mathscr{H}}_m^s) \ni \tilde{\mathcal{S}}_m \Rightarrow \chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(\mathcal{S}) + \underbrace{\frac{1}{2\pi i}\oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^D - \lambda)^{-1}\, d\lambda}_{\text{integral operator with smooth kernel}}.$$

FP states and the mass oscillation property

F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*

$$P = \chi^+(\mathcal{S}) \, \widetilde{G}_m + \text{(smooth)} \qquad \check{P} = \chi^+(\mathcal{S}) \, \check{G}_m + \text{(smooth)} \qquad P^{\text{vac}} = \chi^+(\mathcal{S}) \, G_m$$

- $\omega_2^{\text{vac}}(x, y)$ is Hadamard in Ω_0 and then in $\mathcal{M}$ via the prop. of singularities.
- $\check{\omega}_2(x, y) - \omega_2^{\text{vac}}(x, y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_0$, since $\check{\mathcal{G}}_m(x, y) \equiv \mathcal{G}_m(x, y)$.
- $\check{\omega}_2(x, y)$ is Hadamard in Ω_0 and then in $\mathcal{M}$ via the prop. of singularities.
- $\omega_2(x, y) - \check{\omega}_2(x, y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_1$, since $\widetilde{\mathcal{G}}_m(x, y) \equiv \check{\mathcal{G}}_m(x, y)$.
- $\omega_2(x, y)$ is **Hadamard** in Ω_1 and then in $\mathcal{M}$ via the prop. of singularities.

FP states via the Møller operator (...still work in progress)

[N. Drago, S. M.: "A new class of Fermionic Projectors: Møller operators and mass oscillation properties".]

- Consider $P_V := \Box - V$ and $P_{V'} := \Box - V'$

$$P_{V'} = P_V + V - V' = P_V \big(Id + E_V^{\pm}(V - V') \big)$$

  if $(V - V')$ is compact or future/past compact.

- **Møller operator:** $R_{V',V}^{\pm} := \big(Id + E_V^{\pm}(V - V') \big)^{-1} : \mathcal{S}ol(P_V) \to \mathcal{S}ol(P_{V'})$.

- Fix a Cauchy surface Σ, define $\varrho^+ = 1$ on $J^+(\Sigma_\varepsilon) \setminus \Sigma_\varepsilon$ and $\varrho^+ = 0$ on $J^-(\Sigma_\varepsilon) \setminus \Sigma_\varepsilon$, $\varrho^- = 1 - \varrho^+$ and introduce $m''(m, m') := \varrho^+ m' + m\varrho^-$

$$R_{m'',m}^+ := Id - G_{m''}^{\pm}(m - m'') = Id - G_{m''}^{\pm} \varrho^+ (m' - m)$$
$$R_{m',m''}^- := Id - G_{m'}^{\pm}(m'' - m') = Id - G_{m'}^{\pm} \varrho^- (m - m').$$

- **Ultra Møller operator:** $R_{m',m} := R_{m',m''}^- \circ R_{m'',m}^+ : \mathscr{H}_m \to \mathscr{H}_{m'}$

  new families of solutions: $\quad \psi_m \mapsto \mathfrak{R}\psi_m := (R_{\beta,m}\psi_m)_{\beta \in I}$

FP states via the Møller operator (... still work in progress)

[N. Drago, S. M.: *"A new class of Fermionic Projectors: Møller operators and mass oscillation properties"*.]

- As before $\mathcal{H}_m^s := \overline{(\mathcal{S}ol(\mathcal{D}), (\cdot \mid \cdot)_m^s)}$ and let

$$N(\cdot\,,\cdot) = \,<\mathfrak{p}\mathfrak{R}\,\cdot\,|\mathfrak{p}\mathfrak{R}\,\cdot> : \mathcal{H}_m^s \times \mathcal{H}_m^s \to \mathbb{C}$$

- If the **weak** mass oscillation property holds

$$N(\cdot,\cdot) \leq C(\cdot)\|\cdot\|,$$

then **Fermionic Signature Operator** is $\mathcal{S} \in \mathcal{L}(\mathcal{H}_m^s)$ built out of

$$N(\cdot\,,\cdot) = (\cdot \mid \mathcal{S}\,\cdot)_m^s.$$

- As before, we define the **Fermionic Projector** as

$$\chi^{\pm}(S) := \frac{1}{2|S|}(S \pm |S|).$$

Conclusions

What we know:

- Araki's lemmas: $\omega : \mathcal{F} \to \mathbb{C} \iff Q \in \mathcal{B}(\mathscr{H}_m^s)$;
- Spectral calculus: $\chi^{\pm}(\mathcal{S})\chi^{\pm}(\mathcal{S}) = \chi^{\pm}(\mathcal{S}) \quad \chi^{\pm}(\mathcal{S})\chi^{\mp}(\mathcal{S}) = 0$.

Benefit:

- No use of symmetries;
- Distinguished states.

Price to pay:

- Not always available a sesquilinear form $\implies$ *mass oscillation property*.

Future investigations:

- Hadamard condition.