Cosmological Perturbation Theory and Perturbative Quantum Gravity

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dedicated to Bernard Kay

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Introduction

Big problem for Quantum Gravity: Lack of visible effects

$\Longrightarrow$ Ansätze are tested by consistency, but not by observations.

Consistency requires

- Internal consistency
- $\longrightarrow$ Classical General Relativity
- $\longrightarrow$ Quantum Field Theory on Lorentzian manifolds

At present, none of the existing approaches is known to fulfill these requirements.

Direct approach: perturbative Quantum Gravity

Claim:

- Perturbative quantum gravity is consistent as an effective quantum field theory.
- It reproduces General Relativity and Quantum Field Theory on curved spacetime in appropriate limits.
- In addition, it has already been tested via cosmological perturbation theory in Cosmic Microwave Background.

Problems of perturbative Quantum Gravity:

- Nonrenormalizability
- Existence of local observables?
- What happens with spacetime after quantization?

Tentative answers:

- Renormalization at every order is well defined, hence perturbative Quantum Gravity is an effective field theory whose validity for small energies depends on the size of the new coupling constants occuring in higher orders. In addition there are indications that Quantum Gravity might be asymptotically safe (Reuter et al.).
- Local observables in the sense of relative observables (Rovelli) can be defined (see later).
- Spacetime after quantization is defined in terms of coordinates which are quantum fields.

Quantum Field Theory on curved spacetimes

Plan of the talk: A review of Quantum Field Theory on curved spacetimes including perturbative quantum gravity and comparison with cosmological perturbation theory.

Algebraic quantum field theory is the appropriate framework for quantum field theory on curved spacetime (Kay 1979).

Vacuum state has to be replaced by a distinguished class of states (Hadamard states) (Kay 1983).

Conjecture: All these states are locally quasiequivalent (Kay 1983) (Proof by Verch 1992).

Singularity structure of Hadamard states (Kay and Wald)(1989)

Kay's conjecture: Positivity excludes spacelike singularities (Gonnella-Kay 1989).

Proof by Radzikowski (wave front sets, microlocal spectrum condition)(1993)

Begin of modern QFT on CFT, combining AQFT and microlocal analysis

We start with a globally hyperbolic spacetime $\mathcal{M} = (M, g)$ and illustrate the definition of quantum field theories on $\mathcal{M}$ by the example of a scalar field.

Space of field configurations: $\mathcal{E}(M)$ set of smooth functions

Observables: Functionals $F : \mathcal{E}(M) \to \mathbb{C}$

Dynamics: Lagrangian $\mathcal{L}$

Algebraic structure: For each $\varphi_0 \in \mathcal{E}(M)$ we expand the Lagrangian around φ_0 up to second order and obtain a splitting

$$\mathcal{L}(\varphi_0 + \psi) = \mathcal{L}_0(\psi) + \mathcal{L}_I(\psi)$$

Algebraic structure for the free part:

$\Delta =$ retarded minus advanced Green's function of the field equation

Splitting of Δ:

$$\Delta = 2\mathrm{Im} H$$

H (Hadamard function) bisolution of positive type with one sided wave front set (locally positive frequencies). (On Minkowski space an example is the Wightman 2-point function Δ_+.)

$$\mathrm{WF}\Delta = \{(x,y;k,k'), x,y \in M, k \in T_x^*M, k' \in T_x^*M | (k,k') \neq 0,$$
$$\exists \text{ Nullgeodäte } \gamma \text{ von } x \text{ to } y \text{ with } k,k' \text{ coparallel to } \dot{\gamma} \text{ and}$$
$$k' + P_\gamma k = 0,\ P_\gamma \text{ parallel transport along } \gamma\}$$

$$\mathrm{WF}H = \{(x,y;k,k') \in \mathrm{WF}\Delta | k \in \overline{V_+}\}\ ,$$

Product of observables (Wick's Theorem)

$$(F \star G)(\varphi) = \sum \frac{1}{n!} \langle F^{(n)}(\varphi), H^{\otimes n} G^{(n)}(\varphi) \rangle$$

($F^{(n)}$ nth functional derivative)

Example:

$$\varphi(x) \star \varphi(y) = \varphi(x)\varphi(y) + H(x, y)$$

$$\frac{\varphi(x)^n}{n!} \star \frac{\varphi(y)^m}{m!} = \sum_{k=0}^{\min(n.m)} \frac{\varphi(x)^{(n-k)}}{(n-k)!} \frac{H(x, y)^k}{k!} \frac{\varphi(y)^{(m-k)}}{(m-k)!}$$

(Wick-Theorem)

Time ordering operator (unrenormalized):

$$TF(\varphi) = \sum \frac{1}{2^n n!} \langle H_F^{\otimes n}, F^{(2n)} \rangle$$

$H_F = H + i\Delta_{\mathrm{adv}}$ Feynman propagator associated to H.

Renormalization: Define T on multilocal functionals.

Time ordered product $\cdot_T$

$$F \cdot_T G = T(T^{-1}F \cdot T^{-1}G)$$

$\cdot$ pointwise (classical) product $F \cdot G(\varphi) = F(\varphi)G(\varphi)$

Examples:

$$\varphi(x) \cdot_T \varphi(y) = \varphi(x)\varphi(y) + H_F(x,y)$$

$$\frac{\varphi(x)^2}{2} \cdot_T \frac{\varphi(y)^2}{2} = \frac{\varphi(x)^2}{2}\frac{\varphi(y)^2}{2} + \varphi(x)\varphi(y)H_F(x,y) + \frac{H_F(x,y)^2_{\mathrm{ren}}}{2}$$

Time ordered exponential

$$\exp_T F = T \exp(T^{-1} F)$$

Adding an interaction V (inverse w.r.t. the $\star$-product):

$$R_V(F) = (\exp_T V)^{-1} \star (\exp_T(V) \cdot_T F)$$

Bogoliubov's formula (R_V retarded Mœller map)

$\star$-product of the interacting theory:

$$F \star_V G = R_V^{-1}(R_V(F) \star R_V(G))$$

Full theory obtained by inserting $V = \mathcal{L}_I$.

Perturbative agreement (Hollands-Wald): Theory does not depend on the choice of φ_0 (in the sense of formal power series).

Application to gravity

Configuration space: $\mathcal{E}(M)$ set of globally hyperbolic metrics

Problem: linearized equation of motion not hyperbolic

Solution: gauge fixing via Batalin-Vilkovisky formalism

Algebra of observables constructed as a cohomology class of the BRST operator

Difficulty: Nonexistence of local observables

Solution: Relative observables (Rovelli)

Use physical fields (e.g. curvature scalars) as coordinates

Works on generic backgrounds

Typical observables: $X^a_\Gamma, a = 1, \ldots 4$ scalar fields, local functionals of the configuration $\Gamma = (g, \varphi, \ldots)$ and equivariant, i.e. for a diffeomorphism χ acting on Γ

$$X^a_{\chi^*\Gamma} = X^a_\Gamma \circ \chi \ .$$

Assume that for a given background configuration $\Gamma_0 = (g_0, \varphi_0, \ldots)$ the map

$$X_{\Gamma_0} : x \mapsto (X^1_{\Gamma_0}(x), \ldots, X^4_{\Gamma_0}(x)) \in \mathbb{R}^4$$

is injective.

Then let for Γ near to Γ_0

$$\alpha_\Gamma = X_\Gamma^{-1} \circ X_{\Gamma_0}$$

We then set for any other equivariant scalar field A_Γ

$$\mathcal{A}_\Gamma = A_\Gamma \circ \alpha_\Gamma$$

Thus we obtain gauge invariant fields

$$\mathcal{A}_\Gamma(x) := A_\Gamma(\alpha_\Gamma(x)) \ .$$

Hence gauge invariance is obtained by evaluating the field at a point which is shifted in a Γ-dependent way.

In perturbation theory the observables enter only by their Taylor expansion around the background Γ_0. Up to first order

$$\mathcal{A}_{\Gamma_0+\delta\Gamma} = A_{\Gamma_0} + \langle \frac{\delta A_\Gamma}{\delta\Gamma}(\Gamma_0), \delta\Gamma\rangle + \frac{\partial A_{\Gamma_0}}{\partial x^\mu}\langle \frac{\delta\alpha_\Gamma^\mu}{\delta\Gamma}(\Gamma_0), \delta\Gamma\rangle \ .$$

The last term on the right hand side is necessary in order to get gauge invariant fields (up to 1st order). We find

$$\frac{\delta\alpha_\Gamma^\mu}{\delta\Gamma}(\Gamma_0) = -\left(\left(\frac{\partial X_{\Gamma_0}}{\partial x}\right)^{-1}\right)^\mu_a \frac{\delta X_\Gamma^a}{\delta\Gamma}(\Gamma_0) \ .$$

Observations:

- If A_Γ vanishes on the background, then it is gauge invariant at first order.
- If A_{Γ_0} depends only on 1 variable, the correction involves only the field

$$x_1^\mu = -\left(\left(\frac{\partial X_{\Gamma_0}}{\partial x}\right)^{-1}\right)^\mu_a \langle \frac{\delta X_\Gamma^a}{\delta \Gamma}(\Gamma_0), \delta\Gamma\rangle$$

- If $A_{\Gamma_0} = 0$, the second order correction is

$$2\partial_\mu \langle \frac{\delta A_\Gamma}{\delta \Gamma}(\Gamma_0), \delta\Gamma\rangle \cdot x_1^\mu$$

and involves in general all coordinates.

Application to cosmology

We observe that the expansion of physical observables contains contributions of the physical coordinates expanded around the background.

Inflationary scenario: gravity, coupled to a minimally coupled scalar field φ

Difficulty: background not generic, therefore not sufficiently many physical coordinates

Solution: use φ as time coordinate and add auxiliary fields mimicking fields of the standard model

Toy model: 3 minimally coupled scalar fields X^a, $a = 1, 2, 3$.
Background

$$g_0 = a^2(\tau)(d\tau^2 - d\mathbf{x}^2) \quad , \quad \varphi_0 = \phi \; , \; X_0^a = \epsilon x^a$$

Comparison with cosmological perturbation theory:

$$\delta g = a^2 \begin{pmatrix} -2A & (\partial_i B + V_i)^t \\ -\partial_i B + V_i & 2(\partial_i \partial_j E + \delta_{ij} D + \partial_{(i} W_{j)} + T_{ij}) \end{pmatrix}$$

Interesting observables:

Spatial curvature, defined as curvature of the metric tensor

$$h = g - \frac{d\varphi \otimes d\varphi}{g^{-1}(d\varphi, d\varphi)}$$

On surfaces of constant φ, h is nondegenerate and Riemannian.

Scalar spatial curvature in linear order

$$R_1^{(\varphi)} = \frac{4\mathcal{H}}{\phi'}\Delta\mu$$

$\mathcal{H} = aH$ (conformal Hubble parameter), $\phi' = \frac{d\phi}{d\tau}$,
$\mu = \delta\varphi - \frac{\phi'}{\mathcal{H}}D$ Mukhanov-Sasaki variable

Here no 1st order correction, since the 0th order vanishes.

Lapse function (up to 1st order)

$$N = |g^{-1}(d\varphi, d\varphi)|^{-\frac{1}{2}} = -\frac{a}{\phi'} + \frac{a}{\phi'^2}(\delta\varphi' - A\phi')$$

Correction term

$$\mathcal{N} = N + \frac{a}{\phi'^2}\left(\frac{\phi''}{\phi'} - \mathcal{H}\right)\delta\varphi$$

On shell one obtains

$$\mathcal{N} = -\frac{2a}{\phi'^3}\Delta\Psi$$

Ψ Bardeen potential (analogue of the Newtonian potential)

$$\Psi = A - (\partial_\tau + \mathcal{H})(B + E')$$

Fluctuations in the microwave background are explained by the Sachs-Wolfe effect:

$$\frac{\delta T}{T} = \frac{1}{3}\Psi$$

where Ψ in 1st order is considered as a quantum field.

It involves besides the inflaton field also gravitational degrees of freedom.

Conclusions

- Perturbative quantum gravity provides a consistent picture of quantum fluctuations around a classical background.
- In linear order it reproduces cosmological perturbation theory.
- In principle, computations at every order are possible, but involve (due to the nonrenormalizabilty) in each order a finite number of new parameters which have to fixed by experiment.
- The formulas at higher order involve the used coordinates which should be considered as physical fields . For a realistic computation they should be related to the fields of the standard model.

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