Qualitative analysis of solutions of the semiclassical Einstein equations in FLRW spacetimes

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Table of Contents

Introduction

Geometry of a homogeneous, isotropic spacetime determined by:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right).$$

Dynamics

- **continuity equation:**

  $T_\mu^{\ \mu}(t) := -\rho(t) + 3p(t) = -\left(\frac{1}{H(t)}\frac{d}{dt} + 4\right)\rho(t)$
- **Friedmann equation:**

  $3H^2(t) = 8\pi G\rho(t)$
- **equation of state:**

  $-1 + v(t) = \frac{p(t)}{\rho(t)}$

Example: The classical Λ-CDM model (e.g. in Misner, Thorne, Wheeler 1973)

Spacetime is filled with dust ($v_{dust} = 1$), radiation ($v_{rad} = 4/3$) and dark energy ($v_{de} = 0$).

Use continuity equation to obtain the enegry density of each matter type:

$$\rho^A(t) = \rho_0^A \left(\frac{a(t)}{a_0} \right)^{-3v_A} .$$

Friedmann's equation:

$$H^2(t) = k_1 a^{-4}(t) + k_2 a^{-3}(t) + k_3.$$

For a massless, conformally coupled quantum field in FLRW spacetimes (Barrow, Ottewill 1986 and Hack 2016):

$$\omega(:T^{\mu}_{\mu}:) = \frac{A}{2880\pi^2}\Box R + \frac{B}{2880\pi^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) - r_3 R + r_4,$$

where

$$A = \begin{cases} 1 - 2880\pi^2(3r_1 + r_2) & \text{massless, conformally coupled Scalar field} \\ -18 - 2880\pi^2(3r_1 + r_2) & \text{Maxwell field} \end{cases}$$

and

$$B = 3 \times \begin{cases} 1 & \text{massless, conformally coupled Scalar field} \\ 62 & \text{Maxwell field} \end{cases} .$$

r_i are undetermined renormalisation constants.

Plug $\omega(: T^{\mu}_{\mu} :)$ in the continuity equation to obtain $\rho^{sc}_{\omega}(t)$. Then Friedmann's equation becomes

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 + \frac{1}{6}\frac{B}{A}H^4 - \frac{1}{2}\frac{C}{A}H^2 + \frac{1}{6}\frac{D}{A} + \frac{c_\omega}{a^4},$$

where $C = 360\pi G^{-1}(1 - 8\pi G r_3)$ and $D = -2880\pi^2 r_4$.

General solution can only be found for specific values of renormalisation constants.

Special solutions:

$$H^2_{\pm} = \frac{3}{2}\frac{C}{B}\left(1 \pm \sqrt{1 - \frac{4}{9}\frac{BD}{C^2}}\right).$$

Qualitatively the behavior of all solutions can be studied using dynamical systems theory.

Frequently asked questions:

- Are the Minkowski and deSitter equilibrium stable?
- How “close“ are the solutions to the classical ones?

Use $\dot{H}$ and $\ddot{H}$ to eliminate a in the classical Friedmann equation:

$$H^2(t) = k_1 a^{-4}(t) + k_2 a^{-3}(t) + k_3.$$

becomes

$$0 = \ddot{H}H + 7\dot{H}H^2 + 6H^4 - 6k_3 H^2.$$

Friedmann equation's for classical and quantum matter can be put in the same form:

$$0 = \ddot{H}H + 7\dot{H}H^2 + 6H^4 - 6k_3H^2, \qquad \text{classical}$$
$$0 = \ddot{H}H + 3\dot{H}H^2 + \frac{1}{6}\frac{B}{A}H^4 - \frac{1}{2}\frac{C}{A}H^2 + \frac{1}{6}\frac{D}{A} - \frac{1}{2}\dot{H}^2, \quad \text{semiclassical}.$$

Substitute $v(t) = -\frac{2}{3}\frac{\dot{H}}{H^2}$ then:

$$0 = v'v + H^{-1}\left(c_1v^2 + c_2v + c_3\right) + c_4H^{-3} + c_5H^{-5},$$

where

$$\{c_1, c_2, c_3, c_4, c_5\} = \begin{cases} \{2, -\frac{14}{3}, 4, -\frac{8}{3}k_3, 0\} & \text{classical} \\ \{\frac{3}{2}, -2, \frac{1}{9}\frac{B}{A}, -\frac{2}{9}\frac{C}{A}, -\frac{2}{27}\frac{D}{A}\} & \text{semiclassical} \end{cases}.$$

Dynamical Systems

The Friedmann equation as two dimensional dynamical system reads:

$$\dot{z} = f(z),$$

for $z = (H, \dot{H})^T$ and

$$f(H, \dot{H}) = \begin{pmatrix} \dot{H} \\ (2 - c_1)\frac{\dot{H}^2}{H} + \frac{3}{2}c_2\dot{H}H - \frac{9}{4}c_3H^3 - \frac{9}{4}c_4 - \frac{9}{4}c_5H^{-1} \end{pmatrix}$$

How does the qualitative behavior of trajectories depend on the set of data $\{H_0, \dot{H}_0, c_1, c_2, c_3, c_4, c_5\}$?

$\mathscr{R} : (H, \dot{H}) \mapsto (-H, \dot{H})$ is a reversing symmetry of f, i.e.

$$\frac{d\mathscr{R}(z)}{dt} = -f(\mathscr{R}(z)), \qquad z = (H, \dot{H}).$$

For smooth vector fields f this constrains the set of reversing trajectories: trajectories are crossing $H = 0$ either exactly once or twice or are equilibria lying on $H = 0$ (Vanderbauwhede, Fiedler 1992 and Lamb, Roberts 1998)

Reversing trajectories must cross $(H_0, \dot{H}_0) = (0, \pm\frac{3}{2}\sqrt{\frac{c_5}{2-c_1}}) =: \mathcal{P}_\pm$.

- if $c_5/(2-c_1) < 0$ then there are no reversing trajectories.
- if $c_5/(2-c_1) \geq 0$ there are infinitely many trajectories running through $\mathcal{P}_\pm$.
- if $c_5 = 0$ then the Minkowski equilibrium exists and $\mathcal{P}_+ = \mathcal{P}_- = (0,0)$.

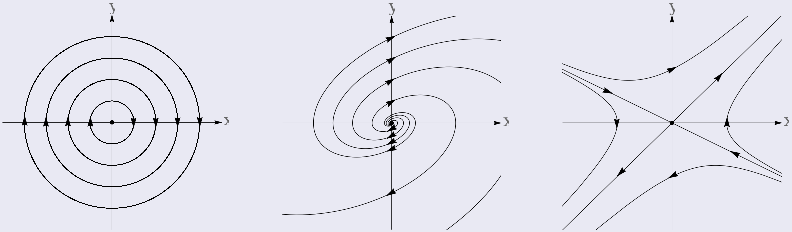
Reversing Solutions are in general not uniquely determined by its inital conditions.

Special choices how to continue solutions reaching $H = 0$: R-symmetric but not smooth or smooth but not R-symmetric.

Stability

Example

$$0 = \ddot{x} + \mu \dot{x} + \nu x$$



centre: $\mu = 0$, $\nu > 0$ — unstable focus: $\mu < 0$, $\nu > 0$ — saddle: $\mu < 0$, $\nu < 0$

Two concepts of stability:

- Lyapunov stability: Do trajectories that are initially close to an equilibrium point remain close for all future times?
- Bifurcation: How does the qualitative behavior of trajectories change when varying the values of involved parameters?

For $H > 0$ and $c_1 \neq 1$ set $x := H^{c_1 - 1}$. Then Friedmann's equation becomes a Liénard type differential equation

$$0 = \ddot{x} + f(x)\dot{x} + g(x),$$

i.e. an equation of an oscillator with non-linear potential $U(x)$ s.t.

$$\frac{dU}{dx} = (c_1 - 1)^{-1} g(x) = \frac{9}{4}\left(c_3 x^{\frac{c_1+1}{c_1-1}} + c_4 x + c_5 x^{\frac{c_1-3}{c_1-1}}\right),$$

and non-linear damping

$$f(x) = -\frac{3}{2} c_2 x^{\frac{1}{c_1-1}}.$$

The stability of equilibrium points can be discussed by using the Energy function $V(x) = 1/2(c_1 - 1)^{-1}\dot{x}^2 + U(x)$ as *Lyapunov function*.

Definition

Two vectofields f and g are called *topologically equivalent* if there is a homeomorphism mapping trajectories of f onto trajectories of g preserving the sense of time.

A bifurcation appears when the topology of a dynamical system changes under variation of its parameters.
Structural Stability: how do *any* "small perturbations" change the "qualitative behavior" of a dynamical system?

Theorem (Andronov, Pontryagin 1937)

$f \in \mathbb{R}^2$ is Structurally stable if and only if

- there is a finite number of equilibrium points and closed trajectories which are all hyperbolic.
- there are no homo- or heteroclinic trajectories.

Consider:

$$0 = \ddot{H}H + (c_1 - 2)\dot{H}^2 - \frac{3}{2}c_2\dot{H}H^2 + \frac{9}{4}c_3H^4 + \frac{9}{4}c_4H^2 + \frac{9}{4}c_5.$$

Bifurcations:

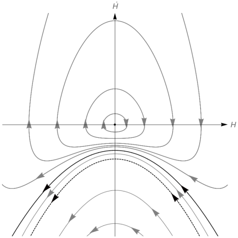
- $c_2 = 0$: if c_2 changes sign trajectories are reversed.
- $c_3 = 0$: $c_3 \to +0$ shifts deSitter equilibria to infinity and for $c_3 < 0$ cease to exist.
- $c_4 = c_5 = 0$: Minkowski equilibrium is non-hyperbolic.
- $c_5 = 0$: Minkowski equilibrium does not exist for $c_5 \neq 0$.
- $c_5 = 4c_4^2/c_3$: deSitter equilibrium is non-hyperbolic.

No bifurcation of $c_1 \in (1, 2]$ in regions where $H \neq 0$.

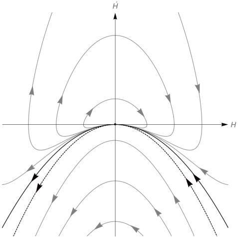
Classical Λ-CDM model: $\{c_1, c_2, c_3, c_4, c_5\} = \{2, -\frac{14}{3}, 4, -\frac{8}{3}k_3, 0\}$

Dynamical equation reduces to Liénard equation:

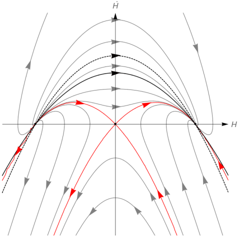
$$0 = \ddot{H}H + 7\dot{H}H^2 + 6H^4 - 6k_3H^2$$



$k_3 < 0$, structurally unstable

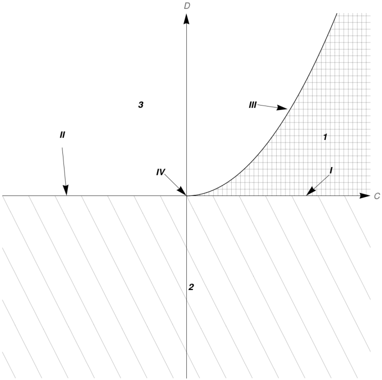


$k_3 = 0$, structurally unstable

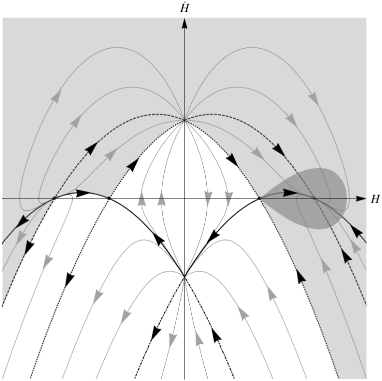


$k_3 > 0$, structurally stable

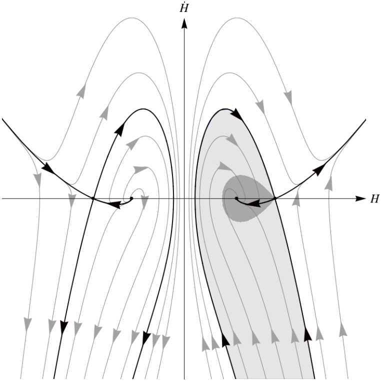
Semiclassical case $\{c_1, c_2, c_3, c_4, c_5\} = \{\frac{3}{2}, -2, \frac{1}{9}\frac{B}{A}, -\frac{2}{9}\frac{C}{A}, -\frac{2}{27}\frac{D}{A}\}$:



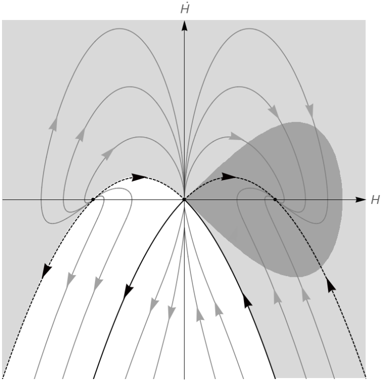
$A = 0$ is a bifurcation: A change of sign turns a saddle into a focus and vice versa.



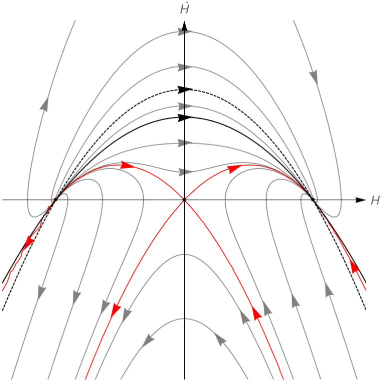
1^+ $(A > 0,\ C > 0,\ 0 < D < \frac{9}{4}\frac{C^2}{B})$



1^- $(A < 0,\ C > 0,\ 0 < D < \frac{9}{4}\frac{C^2}{B})$



I^+ $(A > 0,\ C > 0,\ D = 0)$



Λ-CDM model for $k_3 > 0$

Trace $G_\mu^\mu = 8\pi G\omega(:T_\mu^\mu:)$ of Einsteins field equations determines dynamics for *any* state $\rightarrow$ 3-D dynamical system of variable $z = (H, \dot{H}, \ddot{H})^T$.

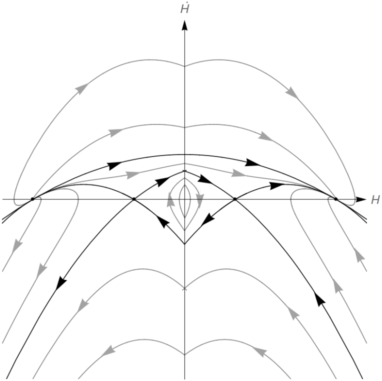
Semiclassical (Vacuum) Friedmann equation determines an invariant manifold (trajectories having inital data on an invariant manifold will remain there for all times).

For special choices of A and D two invariant manifolds $\Sigma_\pm$ determined by

$$0 = \ddot{H} + \left(4H - \frac{k_\pm}{2}\right)\dot{H} + \frac{3}{2}H^3 - \frac{k_\pm}{4}H^2 - \frac{3}{8}k_\pm^2 H + \frac{k_\pm^3}{16},$$

for $k_\pm = \pm\sqrt{\frac{27}{5}\frac{C}{B}}$. For symmetry reasons the smooth continuation at $H = 0$ is not allowed. Symmetric continuation:

$$S := \begin{cases} \Sigma_+ & \text{if } H > 0 \\ \Sigma_- & \text{if } H < 0 \end{cases}, \qquad T := \begin{cases} \Sigma_- & \text{if } H > 0 \\ \Sigma_+ & \text{if } H < 0 \end{cases},$$



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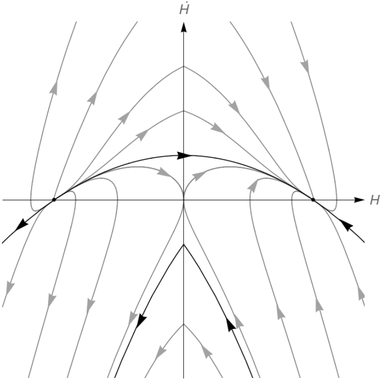
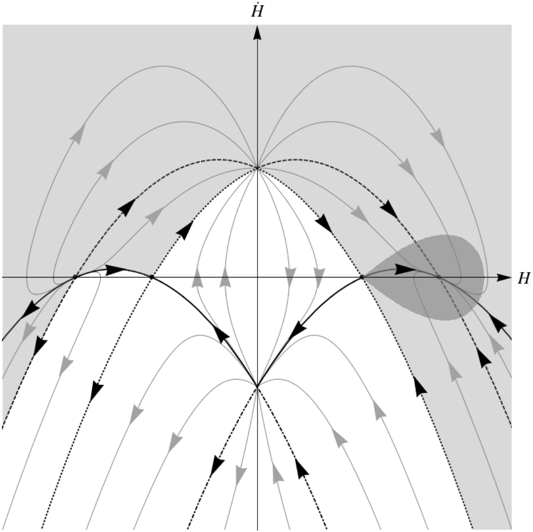


T

Summary

- Classical Λ-CDM model and semiclassical model fulfill same type of DGL.
- The semiclassical model is capable to reproduce most qualitative features of the Λ-CDM model (and other models).
- C acts as cosmological constant and D acts as "second cosmological constant".
- Higher order derivatives stemming from renormalisation pose no a priori problem concerning the qualitative behavior of solutions compared to classical ones (Wald 1977).
- There is a problem concerning smoothness/uniqueness of reversing solutions (Azuma, Wada 1985).
- Qualitative analysis can be used to argue in favour of or against certain values of the renormalisation constants.



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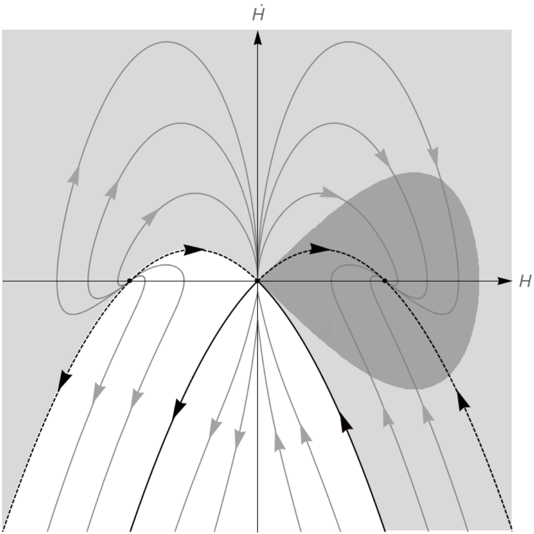

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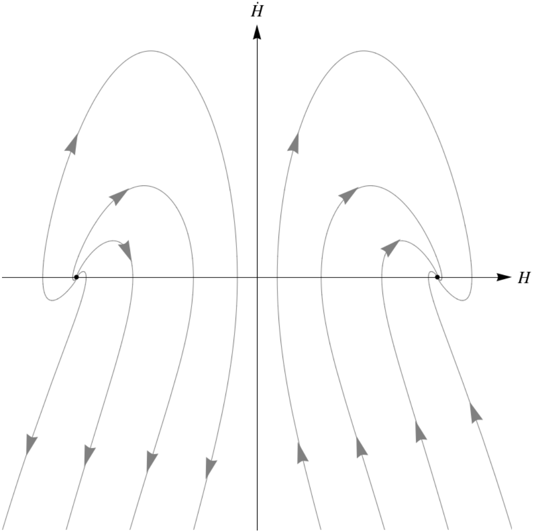

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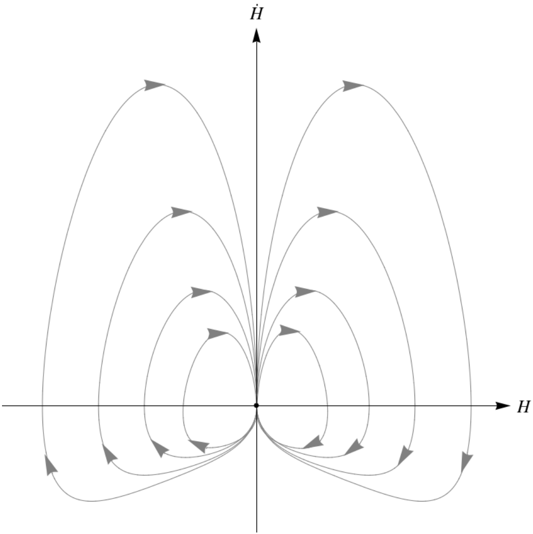

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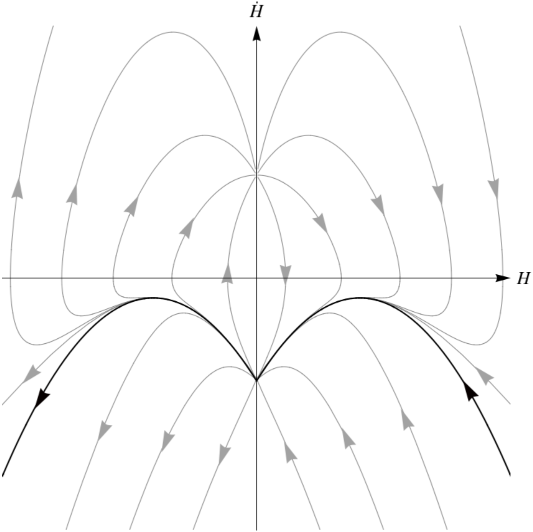

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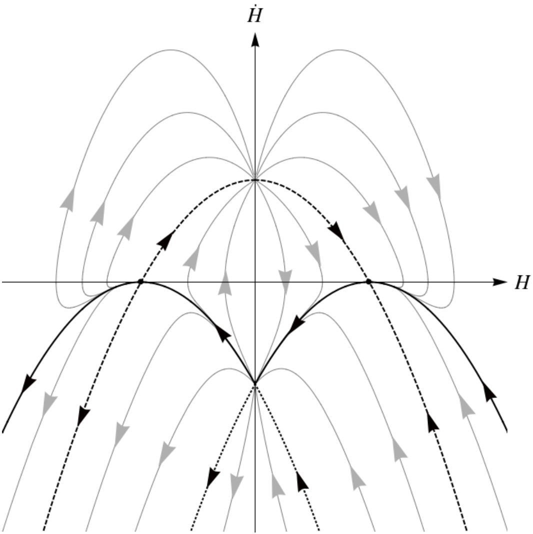

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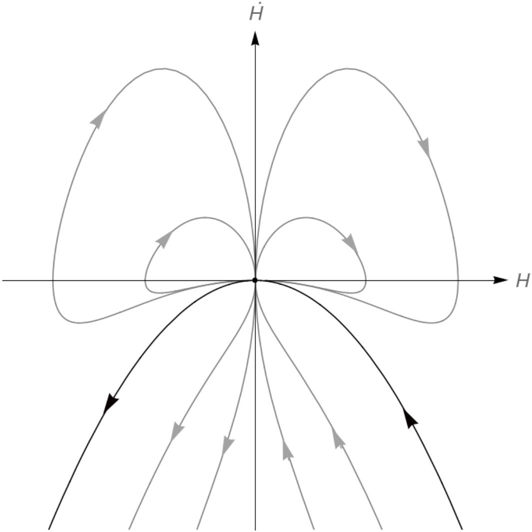

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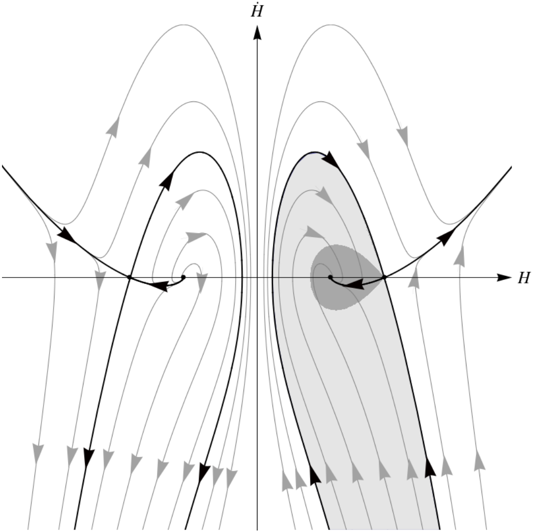

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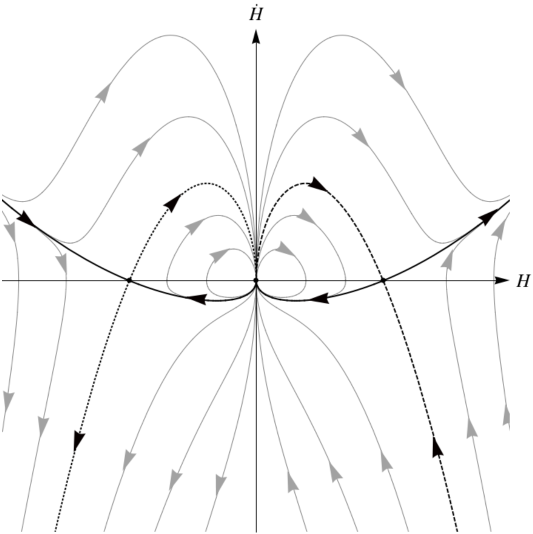

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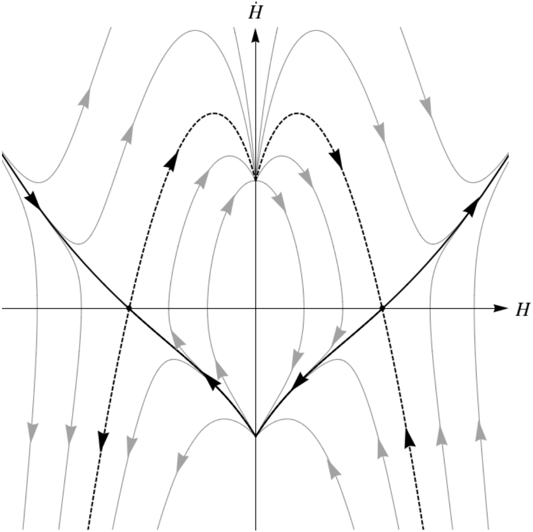

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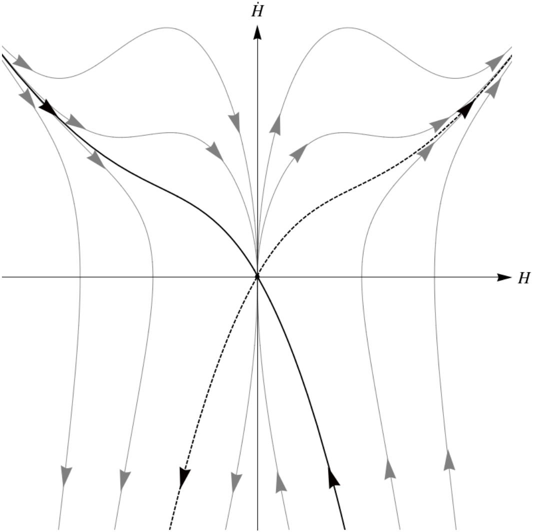

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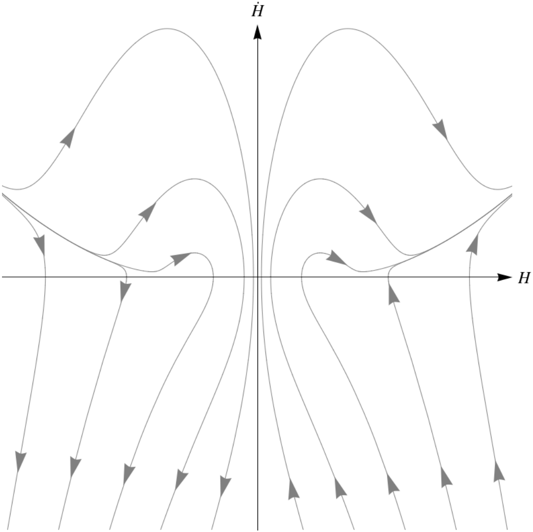

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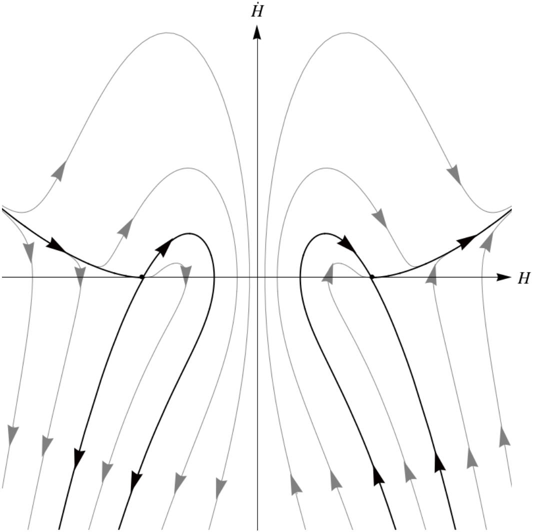

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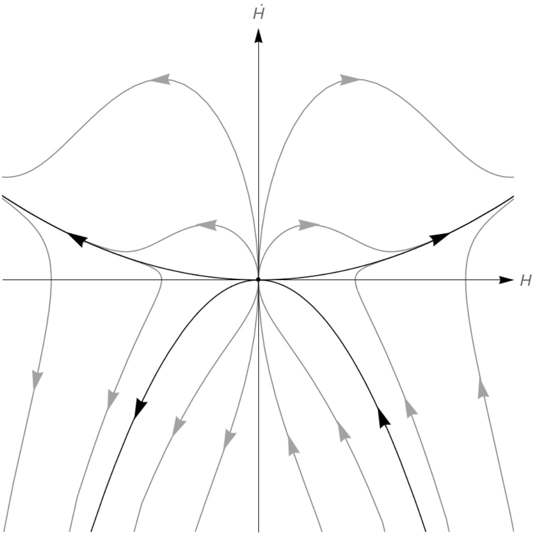

$IV-$

Definition

A set $\mathcal{M}$ is called *positively invariant* if for all $z_0 \in \mathcal{M}$ the positive semitrajectory $\gamma^+(z_0) \subset \mathcal{M}$.
An equilibrium ξ is called *(Lyapunov) stable* if for each neighborhood U of ξ there is a neighborhood $V \subset U$ of ξ which is positively invariant.

Test Lyapunov stability by

- Linearisation: $\dot{z} = Jf(\xi)z$. Local solutions are $z(t) = \xi + \sum_i c_i v_i e^{\lambda_i t}$. Hence the sign of the eigenvalues λ_i of the Jacobian determines the stability.
- Lyapunov function: $V : \mathcal{G} \subset \mathbb{R}^n \to \mathbb{R}$ such that
  (i) $V(\xi) < V(z_0)$ for all $z_0 \in \mathcal{G} \setminus \{\xi\}$
  (ii) $\dot{V}(z(t)) = \langle \nabla V, f \rangle \leq 0$ for all $z_0 \in \mathcal{G} \setminus \{\xi\}$

  Then ξ is asymptotically stable.

How do "small perturbations" change the "qualitative behavior" of a dynamical system?

"small perturbations"

Two vectorfields f and g are called C^k-*close* if $\|f - g\|_k \leq \epsilon$ for some $\epsilon > 0$ in the C^k-norm

$$\|f\|_k := \sup_{x \in \mathcal{G}} \left\{ \sum_{r=0}^{k} \|D^r f(x)\| \right\}$$

"same qualitative behavior"

Two vectorfields f and g are said to be *topologically equivalent* if there is a homeomorphism mapping trajectories of f onto trajectories of g preserving the sense of time.

Structural stability

A dynamical system is called *structurally stable* if any sufficiently C^1-close vectorfield g is topologically equivalent to f.

In two dimensions we have (Andronov/Pontryagin):

Theorem

f is Structurally stable if and only if

- there is a finite number of equilibrium points and closed trajectories which are all hyperbolic.
- there are no homo- or heteroclinic trajectories.

If the vectorfield depends on parameters $\mu \in \mathbb{R}^m$ then $\mu = \mu_0$ is a bifurcation value if $f(x, \mu_0)$ is structurally unstable.

There are other classical matter models that lead to similar dynamical equations, e.g.:

1. The generalised Chaplygin gas (equation of state $p = -A\rho^{-\alpha}$):

$$0 = \ddot{H}H + 2\alpha\dot{H}^2 + 3(1+\alpha)\dot{H}H^2.$$

2. Certain $f(R)$-theories (when $f(R) = -2\Lambda + R - \frac{1}{6}\alpha R^2$):

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 - \alpha^{-1}H^2 + \frac{\Lambda}{\alpha}.$$

3. Imperfect fluid:

$$0 = \ddot{H}H + \zeta\dot{H}^2 + \eta\dot{H}H^2 + \lambda_4 H^4.$$

$$0 = \ddot{H}H + \zeta\dot{H}^2 + \eta\dot{H}H^2 + \lambda_4 H^4 + \lambda_2 H^2 + \lambda_0.$$

seems to be a quiet general cosmological model.