On the algebraic quantization of a scalar field in anti-de Sitter spacetime

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Joint work with Claudio Dappiaggi

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0. Introduction

- We discuss the algebraic quantization of a Klein-Gordon field in anti-de Sitter (AdS), a simple example of a non-globally hyperbolic spacetime, extending the work of Avis, Isham, Storey (1978), Allen & Jacobson (1986) and others.
- We consider Robin boundary conditions at infinity, by treating the system as a Sturm-Liouville problem, complementing the work of Wald & Ishibashi (2004).
- We show that it is possible to associate an algebra of observables enjoying the standard properties of causality, time-slice axiom and F-locality.
- We characterize the wavefront set of the ground state and propose a natural generalization of the definition of Hadamard states in AdS.

Outline

1. Anti-de Sitter spacetime

- **Definition:** *Anti-de Sitter* AdS_{d+1} $(d \geq 2)$ is the maximally symmetric solution to the vacuum Einstein's equations with a negative cosmological constant $\Lambda < 0$.

  It is defined as the hypersurface in $\mathbb{R}^{d+2}$ with line element

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_1^2 + \sum_{i=2}^{d+1} \mathrm{d}X_i^2$$

  given by the relation

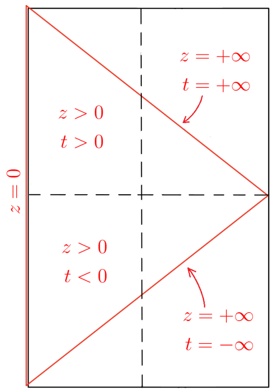
$$-X_0^2 - X_1^2 + \sum_{i=2}^{d+1} X_i^2 = -\ell^2 \,, \qquad \ell \doteq -\frac{d(d-1)}{\Lambda} \,.$$

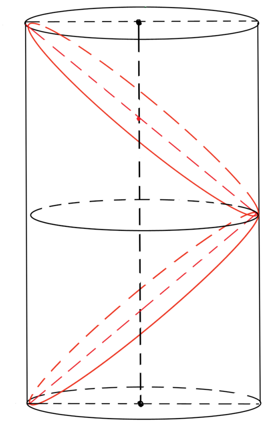
- Anti-de Sitter spacetime is *not* globally hyperbolic: it possesses a timelike boundary at spatial infinity.

- Poincaré patch (t, z, x_i), $t \in \mathbb{R}$, $z \in \mathbb{R}_{>0}$ and $x_i \in \mathbb{R}$, $i = 1, \ldots, d-1$,

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \delta^{ij}\mathrm{d}x_i \mathrm{d}x_j\right) \ .$$

The region covered by this chart is the *Poincaré fundamental domain*, PAdS_{d+1}.

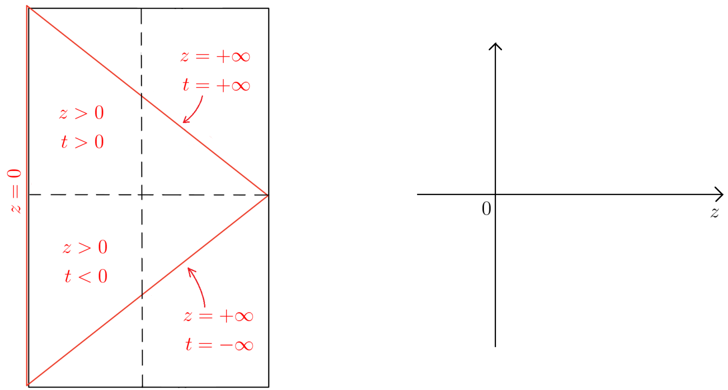




- PAdS_{d+1} can be mapped to $\mathring{\mathbb{H}}^{d+1} \doteq \mathbb{R}_{>0} \times \mathbb{R}^d$ via a conformal rescaling

$$\mathrm{d}s^2 \mapsto \frac{z^2}{\ell^2}\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \delta^{ij}\mathrm{d}x_i\mathrm{d}x_j \,.$$

We can attach a conformal boundary as the locus $z = 0$ and obtain $\mathbb{H}^{d+1} \doteq \mathbb{R}_{\geq 0} \times \mathbb{R}^d$, the *half Minkowski spacetime.*



- **Remark:** From now on, we set $\ell = 1$.

Outline

2.1. Klein-Gordon equation and boundary conditions

- **Klein-Gordon equation.** Poincaré domain (PAdS_{d+1}, g), $\phi : \mathrm{PAdS}_{d+1} \to \mathbb{R}$,

$$P\phi = \left(\Box_g - m_0^2 - \xi R\right)\phi = 0\,.$$

- **Lemma:** *In* $(\mathring{\mathbb{H}}^{d+1}, \eta)$, $\Phi = z^{\frac{1-d}{2}}\phi : \mathring{\mathbb{H}}^{d+1} \to \mathbb{R}$ *is a solution of*

$$P_\eta \Phi = \left(\Box_\eta - \frac{m^2}{z^2}\right)\Phi = 0\,,$$

with $m^2 \doteq m_0^2 - (\xi - \frac{d-1}{4d})R$.

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with $m^2 \doteq m_0^2 - (\xi - \frac{d-1}{4d})R$.

- **Fourier expansion.** Fourier representation of Φ:

$$\Phi = \int_{\mathbb{R}^d} \mathrm{d}^d\underline{k}\, e^{i\underline{k}\cdot\underline{x}}\, \widehat{\Phi}_{\underline{k}}\,, \qquad \underline{x} \doteq (t, x_1, \ldots, x_{d-1})\,, \quad \underline{k} \doteq (\omega, k_1, \ldots, k_{d-1})\,,$$

where $\widehat{\Phi}_{\underline{k}}$ are solutions of the ODE

$$L\widehat{\Phi}_{\underline{k}}(z) \doteq \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{m^2}{z^2}\right)\widehat{\Phi}_{\underline{k}}(z) = \lambda\,\widehat{\Phi}_{\underline{k}}(z)\,, \qquad \lambda \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$$

This is a *Sturm-Liouville problem* on $z \in (0, +\infty)$ with spectral parameter λ.

2.1. Klein-Gordon equation and boundary conditions

- **Definition:** For any $z_0 \in (0, \infty)$ we call *maximal domain* associated to L

$$D_{\max}(L; z_0) \doteq \left\{ \Psi : (0, z_0) \to \mathbb{C} \mid \Psi, \tfrac{\mathrm{d}\Psi}{\mathrm{d}z} \in AC_{\mathrm{loc}}(0, z_0) \text{ and } \Psi, L(\Psi) \in L^2(0, z_0) \right\},$$

  where $AC_{\mathrm{loc}}(0, z_0)$ is the collection of all complex-valued, locally absolutely continuous functions on $(0, z_0)$.

- Fundamental pair of solutions of $L\Phi = \lambda\Phi \doteq q^2\Phi$ as

$$\Phi_1(z) = \sqrt{\frac{\pi}{2}} q^{-\nu} \sqrt{z}\, J_\nu(qz),$$

$$\Phi_2(z) = \begin{cases} -\sqrt{\dfrac{\pi}{2}}\, q^\nu \sqrt{z}\, J_{-\nu}(qz), & \nu \in (0, 1), \\ -\sqrt{\dfrac{\pi}{2}} \sqrt{z} \left[Y_0(qz) - \dfrac{2}{\pi} \log(q) \right], & \nu = 0, \end{cases}$$

  where $\nu \doteq \frac{1}{2}\sqrt{1 + 4m^2} \geq 0$. Only $\Phi_1 \in L^2(0, z_0)$ for $\nu \geq 1$.

2.1. Klein-Gordon equation and boundary conditions

- **Definition:** $\Psi_\alpha : (0, \infty) \to \mathbb{C}$ satisfies an *α-boundary condition* at the endpoint 0, or equivalently that $\Psi_\alpha \in D_{\max}(L; \alpha)$, if the following two conditions are satisfied:
  1. there exists $z_0 \in (0, \infty)$ such that $\Psi_\alpha \in D_{\max}(L; z_0)$;
  2. there exists $\alpha \in (0, \pi]$ such that

$$\lim_{z \to 0} \left\{ \cos(\alpha)\, W_z[\Psi_\alpha, \Phi_1] + \sin(\alpha)\, W_z[\Psi_\alpha, \Phi_2] \right\} = 0 \,,$$

  where $W_z[\Psi_\alpha, \Phi_i] \doteq \Psi_\alpha \frac{\mathrm{d}\Phi_i}{\mathrm{d}z} - \Phi_i \frac{\mathrm{d}\Psi_\alpha}{\mathrm{d}z}$, $i = 1, 2$, is the Wronskian.

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- If $\nu \in [0,1)$, Ψ_α may then be written as

$$\Psi_\alpha = \cos(\alpha)\Phi_1 + \sin(\alpha)\Phi_2\,.$$

  If $\nu \geq 1$, no boundary conditions at $z = 0$ are imposed and we may take $\Psi \equiv \Psi_\pi$.

- These boundary conditions are also commonly known as *Robin boundary conditions*.

2.2. Causal propagator

- The building block necessary for the algebraic quantization is the *causal propagator* $G_\alpha \in \mathcal{D}'(\mathrm{PAdS}_{d+1} \times \mathrm{PAdS}_{d+1})$. The propagator in PAdS_{d+1} can be reconstructed via

$$G_\alpha = (zz')^{\frac{d-1}{2}} G_{\mathbb{H},\alpha} \,.$$

with $G_{\mathbb{H},\alpha} \in \mathcal{D}'\big(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1}\big)$. The latter satisfies

$$(P_\eta \otimes \mathbb{I})\, G_{\mathbb{H},\alpha} = (\mathbb{I} \otimes P_\eta)\, G_{\mathbb{H},\alpha} = 0 \,,$$
$$G_{\mathbb{H},\alpha}(f, f') = -G_{\mathbb{H},\alpha}(f', f) \quad \forall f, f' \in C_0^\infty\big(\mathring{\mathbb{H}}^{d+1}\big) \,,$$
$$G_{\mathbb{H},\alpha}|_{t=t'} = 0 \,, \quad \partial_t G_{\mathbb{H},\alpha}|_{t=t'} = \partial_{t'} G_{\mathbb{H},\alpha}|_{t=t'} = \prod_{i=1}^{d-1} \delta(x_i - x_i')\delta(z - z') \,.$$

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- We consider a mode expansion for the integral kernel of $G_{\mathbb{H},\alpha}$,

$$G_{\mathbb{H},\alpha}(\underline{x} - \underline{x}', z, z') = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d \underline{k}}{(2\pi)^{\frac{d}{2}}} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\, \widehat{G}_{\underline{k},\alpha}(z, z') \,,$$

where $\widehat{G}_{\underline{k},\alpha}(z, z')$ is a symmetric solution of

$$(L \otimes \mathbb{I})\widehat{G}_{\underline{k},\alpha}(z, z') = (\mathbb{I} \otimes L)\widehat{G}_{\underline{k},\alpha}(z, z') = q^2\, \widehat{G}_{\underline{k},\alpha}(z, z') \,, \quad L \doteq -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{m^2}{z^2} \,, \quad q^2 = \underline{k} \cdot \underline{k} \,.$$

2.2. Causal propagator

Let $r^2 \doteq \sum_{i=1}^{d-1} \left(x^i - x'^i\right)^2$ and

$$I_\epsilon(q,r,t,t') \doteq \int_0^\infty \mathrm{d}k\, k \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\, q\, \frac{\sin\left(\sqrt{k^2+q^2}(t-t'-i\epsilon)\right)}{\sqrt{2\pi(k^2+q^2)}} .$$

Proposition: *The causal propagator $G_{\mathbb{H},\alpha} \in \mathcal{D}'\left(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1}\right)$ for different values of $\nu \in [0,\infty)$ has integral kernel given by the following expressions.*

1. *If $\nu \in [1,\infty)$,*

$$G_{\mathbb{H},\pi}(x,x') = \lim_{\epsilon\to 0^+} \sqrt{zz'} \int_0^\infty \mathrm{d}q\, I_\epsilon(q,r,t,t')\, J_\nu(qz) J_\nu(qz') .$$

2. *If $\nu \in (0,1)$ and $c_\alpha \doteq \cot(\alpha) \leq 0$, that is, $\alpha \in [\frac{\pi}{2},\pi]$,*

$$G_{\mathbb{H},\alpha}(x,x') = \lim_{\epsilon\to 0^+} \sqrt{zz'} \int_0^\infty \mathrm{d}q\, I_\epsilon(q,r,t,t')\, \frac{\psi_{c_\alpha}(z)\psi_{c_\alpha}(z')}{c_\alpha^2 - 2c_\alpha q^{2\nu}\cos(\nu\pi) + q^{4\nu}} ,$$

where $\psi_{c_\alpha}(z) = c_\alpha J_\nu(qz) - q^{2\nu} J_{-\nu}(qz)$.

Remark: There is **no** ground state for Robin boundary conditions with $c > 0$ and for $\nu = 0$, so these cases will not be further considered.

2.2. Causal propagator

Proposition: *Let:*

$$G^{(\mathrm{D})}(x,x') = \lim_{\epsilon\to 0^+}\left[\frac{F\left(\frac{d}{2}+\nu, \frac{1}{2}+\nu; 1+2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}+\nu}} - (\epsilon \leftrightarrow -\epsilon)\right],$$

$$G^{(\mathrm{N})}(x,x') = \lim_{\epsilon\to 0^+}\left[\frac{F\left(\frac{d}{2}-\nu, \frac{1}{2}-\nu; 1-2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}-\nu}} - (\epsilon \leftrightarrow -\epsilon)\right],$$

where $\sigma_\epsilon \doteq \sigma + 2i\epsilon(t-t') + \epsilon^2$ *and* F *is the Gaussian hypergeometric function.*

The integral kernel of the causal propagator on $PAdS_{d+1}$ *is*

$$G_\alpha(u) = \mathcal{N}_\alpha\left[\cos(\alpha)\, G^{(\mathrm{D})}(u) + \sin(\alpha)\, G^{(\mathrm{N})}(u)\right]$$

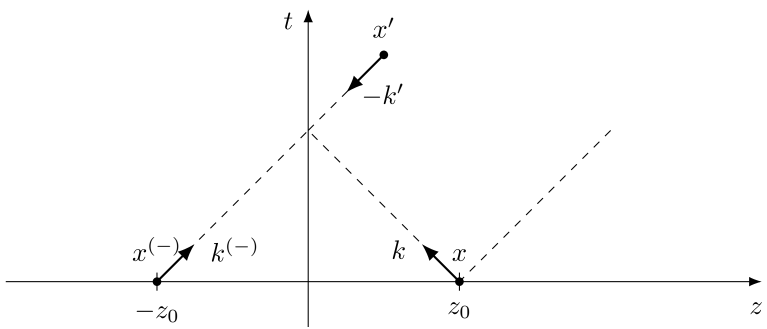
where $\mathcal{N}_\alpha$ *is a normalization constant,* $\nu \in (0,1)$ *and* $\alpha \in [\frac{\pi}{2}, \pi]$.

2.2. Causal propagator

Theorem: *The wavefront set of the causal propagator $G_{\mathbb{H},\alpha}$ in $\mathring{\mathbb{H}}^{d+1}$ is given by*

$$WF(G_{\mathbb{H},\alpha}) = \left\{(x,k;x',k') \in T^*(\mathring{\mathbb{H}}^{d+1})^{\times 2} \setminus \{0\} : (x,k) \sim_{\pm} (x',k')\right\}$$

- $\sim_{\pm}$: *∃ null geodesics* $\gamma, \gamma^{(-)} : [0,1] \to \mathring{\mathbb{H}}^{d+1}$ *with*
  - $\gamma(0) = x = (\underline{x}, z)$, $\gamma^{(-)}(0) = x^{(-)} = (\underline{x}, -z)$ *and* $\gamma(1) = x'$;
  - $k = (k_{\underline{x}}, k_z)$ $(k^{(-)} = (k_{\underline{x}}, -k_z))$ *is coparallel to* γ $(\gamma^{(-)})$ *at 0;*
  - $-k'$ *is the parallel transport of* k $(k^{(-)})$ *along* γ $(\gamma^{(-)})$ *at 1.*



Remark: These results are in full agreement with Wrochna (2016) for $\alpha = \pi$.

2.2. Causal propagator

Definition: We call space of *off-shell configurations* with an α-boundary condition

$$\mathcal{C}_\alpha\left(\mathring{\mathbb{H}}^{d+1}\right) \doteq \left\{\Phi_\alpha \in C^\infty\left(\mathring{\mathbb{H}}^{d+1}\right) \mid \widehat{\Phi}_{\underline{k},\alpha} \in D_{\max}(L;\alpha)\right\} ,$$

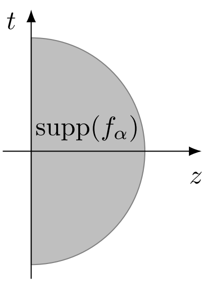
where

$$\widehat{\Phi}_{\underline{k},\alpha} \equiv \widehat{\Phi}_{\underline{k},\alpha}(z) = \int_{\mathbb{R}^n} \frac{\mathrm{d}^d \underline{x}}{(2\pi)^{\frac{d}{2}}} \, e^{-i\underline{k}\cdot_d \underline{x}} \, \Phi_\alpha(\underline{x}, z) ,$$

and $\underline{x} = (t, x_1, ..., x_{d-1})$ and $\underline{k} = (\omega, k_1, ..., k_{d-1})$.

Secondly, we define

$$\begin{aligned} \tilde{\mathcal{C}}_{\alpha,0}\left(\mathring{\mathbb{H}}^{d+1}\right) \doteq \Big\{\Phi_\alpha \in \mathcal{C}_\alpha\left(\mathring{\mathbb{H}}^{d+1}\right) \mid \exists\, F_1, F_2 \in C_0^\infty\left(\mathbb{H}^{d+1}\right) \\ \Phi_\alpha(\underline{x}, z) = \cos(\alpha) z^{\nu+\frac{1}{2}} F_1(\underline{x}, z) \\ + \sin(\alpha) z^{-\nu+\frac{1}{2}} F_2(\underline{x}, z) \Big\} . \end{aligned}$$



$f_\alpha \in \tilde{\mathcal{C}}_{\alpha,0}\left(\mathring{\mathbb{H}}^{d+1}\right) \setminus C_0^\infty\left(\mathring{\mathbb{H}}^{d+1}\right)$

Proposition:

$$\ker(G_{\mathbb{H},\alpha})\big|_{\tilde{\mathcal{C}}_{\alpha,0}(\mathring{\mathbb{H}}^{d+1})} = P_\eta\left[\widetilde{C}_{\alpha,0}\left(\mathring{\mathbb{H}}^{d+1}\right)\right] .$$

Outline

3.1. Algebra of observables

Definition: We call $\mathcal{A}_\alpha(\mathring{\mathbb{H}}^{d+1})$ the *off-shell* $*$*-algebra* of the system with complex conjugation as $*$-operation. It is generated by the functionals

$$F_{f_\alpha}(\phi) = \int_{\mathring{\mathbb{H}}^{d+1}} \mathrm{d}^{d+1}x\, \phi_\alpha(x) f_\alpha(x)\,,$$

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Definition: We call $\mathcal{A}^{\mathrm{on}}_\alpha(\mathring{\mathbb{H}}^{d+1})$ the *on-shell* $*$*-algebra* of the system, generated by the functionals $F_{[f_\alpha]}$, with $[f_\alpha] \in \frac{\tilde{\mathcal{C}}_{\alpha,0}(\mathring{\mathbb{H}}^{d+1})}{P_\eta[\tilde{\mathcal{C}}_{\alpha,0}(\mathring{\mathbb{H}}^{d+1})]}$ and

$$F_{[f_\alpha]}(\phi_\alpha) = \int_{\mathring{\mathbb{H}}^{d+1}} \mathrm{d}^{d+1}x \, f_\alpha(x)\phi_\alpha(x) \,.$$

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Definition: We call $\mathcal{A}_\alpha^{\mathrm{on}}(\mathring{\mathbb{H}}^{d+1})$ the *on-shell $*$-algebra* of the system, generated by the functionals $F_{[f_\alpha]}$, with $[f_\alpha] \in \frac{\tilde{\mathcal{C}}_{\alpha,0}(\mathring{\mathbb{H}}^{d+1})}{P_\eta[\tilde{\mathcal{C}}_{\alpha,0}(\mathring{\mathbb{H}}^{d+1})]}$ and

$$F_{[f_\alpha]}(\phi_\alpha) = \int_{\mathring{\mathbb{H}}^{d+1}} \mathrm{d}^{d+1}x\, f_\alpha(x)\phi_\alpha(x)\,.$$

Proposition: *The algebra $\mathcal{A}_\alpha^{\mathrm{on}}(\mathring{\mathbb{H}}^{d+1})$ is*

1. causal, *that is, algebra elements supported in spacelike separated regions commute.*
2. *fulfils the* time-slice axiom, *i.e. let $O_{\epsilon,\bar{t}} \doteq (\bar{t}-\epsilon, \bar{t}+\epsilon) \times \mathring{\mathbb{H}}^d$, $\epsilon > 0$ and $\bar{t} \in \mathbb{R}$, and let $\mathcal{A}_\alpha^{\mathrm{on}}(O_{\epsilon,\bar{t}})$ be the on-shell algebra restricted to $O_{\epsilon,\bar{t}}$, then $\mathcal{A}_\alpha^{\mathrm{on}}(\mathring{\mathbb{H}}^{d+1}) \simeq \mathcal{A}_\alpha^{\mathrm{on}}(O_{\epsilon,\bar{t}})$.*
3. *is* F-local, *namely it is $*$-isomorphic to $\mathcal{A}(D)$, where D is any globally hyperbolic subregion of $\mathring{\mathbb{H}}^{d+1}$.*

3.2. Hadamard condition in AdS

Proposition: *Let:*

$$\omega_2^{(\mathrm{D})}(x,x') = \lim_{\epsilon\to 0^+} \frac{F\left(\frac{d}{2}+\nu, \frac{1}{2}+\nu; 1+2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}+\nu}},$$

$$\omega_2^{(\mathrm{N})}(x,x') = \lim_{\epsilon\to 0^+} \frac{F\left(\frac{d}{2}-\nu, \frac{1}{2}-\nu; 1-2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}-\nu}},$$

where $\sigma_\epsilon \doteq \sigma + 2i\epsilon(t-t') + \epsilon^2$ *and* F *is the Gaussian hypergeometric function.*

The integral kernel of the two-point function associated with the ground state is

$$\omega_{2,\alpha}(u) = \mathcal{N}_\alpha \left[\cos(\alpha)\,\omega_2^{(\mathrm{D})}(u) + \sin(\alpha)\,\omega_2^{(\mathrm{N})}(u)\right]$$

where $\mathcal{N}_\alpha$ *is a normalization constant,* $\nu \in (0,1)$ *and* $\alpha \in [\frac{\pi}{2}, \pi]$.

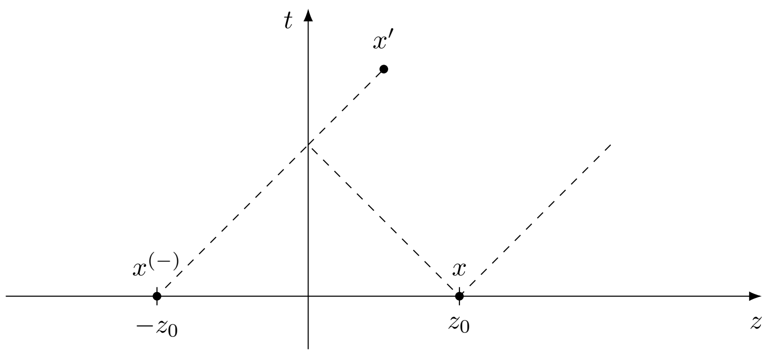
3.2. Hadamard condition in AdS

Proposition: *Let $H(x,x')$ be the Hadamard parametrix in $PAdS_{d+1}$ and let $H^{(-)}(x,x') \doteq \iota_z H(x,x')$, where $\iota_z(x,x') \doteq (\underline{x}, -z; \underline{x}', z')$. Then, if $\alpha \neq \frac{3\pi}{4}$, the two-point distribution $\omega_{2,\alpha}(x,x')$ is such that*

$$\omega_{2,\alpha}(x,x') - H(x,x') - i(-1)^{-\nu}\frac{\cos(\alpha) + (-1)^{-2\nu}\sin(\alpha)}{\cos(\alpha)+\sin(\alpha)} H^{(-)}(x,x')$$

lies in $C^\infty(PAdS_{d+1} \times PAdS_{d+1})$.

Remark: If $\nu = \frac{1}{2}$, we recover the *method of images.*

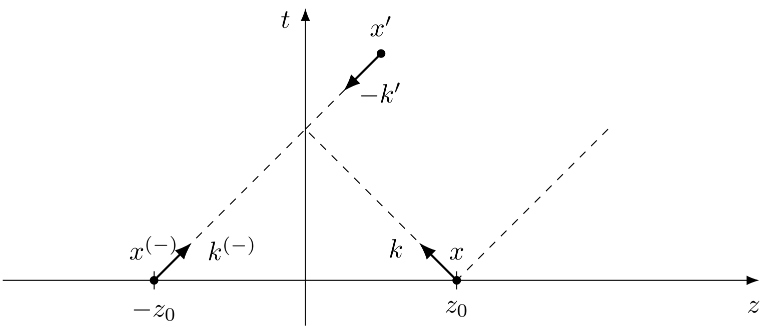


3.2. Hadamard condition in AdS

Theorem: *The wavefront set of the two-point distribution* $\omega_{2,\alpha}^{\mathbb{H}}$ *in* $\mathring{\mathbb{H}}^{d+1}$ *is given by*

$$WF(\omega_{2,\alpha}^{\mathbb{H}}) = \left\{(x,k;x',k') \in T^*(\mathring{\mathbb{H}}^{d+1})^{\times 2} \setminus \{0\} : (x,k) \sim_{\pm} (x',k'),\ k \triangleright 0\right\}$$

- $\sim_{\pm}$: $\exists$ *null geodesics* $\gamma, \gamma^{(-)} : [0,1] \to \mathring{\mathbb{H}}^{d+1}$ *with*
  - $\gamma(0) = x = (\underline{x}, z)$, $\gamma^{(-)}(0) = x^{(-)} = (\underline{x}, -z)$ *and* $\gamma(1) = x'$;
  - $k = (k_{\underline{x}}, k_z)$ ($k^{(-)} = (k_{\underline{x}}, -k_z)$) *is coparallel to* γ ($\gamma^{(-)}$) *at 0;*
  - $-k'$ *is the parallel transport of* k ($k^{(-)}$) *along* γ ($\gamma^{(-)}$) *at 1;*
- $k \triangleright 0$: k *is future-directed.*



3.2. Hadamard condition in AdS

Definition: We call a state $\omega^{\mathbb{H}}$ a **Hadamard state** for a scalar field in $\mathring{\mathbb{H}}^{d+1}$ if its two-point function has a wavefront set as above.

This definition can be read as a generalization at the level of states of F-locality.

Proposition: Any Hadamard state ω for a scalar field on $\mathring{\mathbb{H}}^{d+1}$ is such that $\omega_{2,D}$, the restriction to any globally hyperbolic subregion $D \subset \mathring{\mathbb{H}}^{d+1}$ of the two-point function $\omega_2^{\mathbb{H}}$, has a wavefront set of Hadamard form

$$WF(\omega_{2,D}) = \left\{(x,k;x',k') \in T^*(D \times D) \setminus \{0\} : (x,k) \sim (x',k'),\, k \triangleright 0\right\}.$$

Remarks:

- These results are in full agreement with Wrochna (2016).
- With the definition of Hadamard states above, it is possible to construct a global algebra of Wick polynomials in AdS.

Outline

4. Conclusions

- We analysed the algebraic quantization of a real, massive scalar field in AdS in terms of a equivalent theory in $\mathring{\mathbb{H}}^{d+1}$. We treated the classical dynamics as a singular Sturm-Liouville problem and considered Robin boundary conditions at infinity, which only depend on the mass of the field.

- We computed the two-point function for the ground state obeying these Robin boundary conditions and obtained its wavefront set. Besides the usual singularity along null geodesics, there exists only one extra singularity along reflected null geodesics, independently of the mass of the field. This suggests a natural generalization of the Hadamard condition to spacetimes with timelike boundaries.

- Ongoing work:
  - extend this formalism to stationary, asymptotically AdS spacetimes;
  - construct Hadamard states in other similar spacetimes, such as AdS black holes (ongoing: BTZ black hole).

THANK YOU FOR YOUR ATTENTION!