Invariant states on Weyl algebras for the action of the symplectic group

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AQFT: WHERE OPERATOR ALGEBRA MEETS MICROLOCAL ANALYSIS

Cortona, 8th of June 2018

Joint project with Federico Bambozzi and Nicola Pinamonti

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♡ GOAL: CLASSIFY $Sp(2g,\mathbb{Z})$-INVARIANT STATES ON WEYL ALGEBRAS

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- Since $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^{2g}$ by $m = (m_1, m_2) \mapsto \widetilde{m} = (m_1, 0 \dots, 0, m_2, 0, \dots, 0)$, we set $g = 1$.

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$Sp(2, \mathbb{Z})$-invariant states

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τ is obviously invariant: $\tau(\Phi_\Theta W_m) = \tau(W_{\theta m}) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$

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(5) Fix $\chi \neq 0$.

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(3) Choosing $\mathfrak{a} = W_0 + W_\chi \xRightarrow{\omega(\mathfrak{a}^*\mathfrak{a})\geq 0} 1 - p^2 \leq 0$

(4) **Hence, any $Sp(2,\mathbb{Z})$-invariant states reads as**

$$\omega(W_\chi) = \begin{cases} 1 & \text{if } \chi = (0,0) \\ p^{(\chi)} \in [-1,1] & \text{else} \end{cases}$$

Next, we choose a more suitable χ without loosing of generality

(5) Fix $\chi \neq 0$.

(6) By previous Prop.: For any χ there exists $\Theta \in Sp(2,\mathbb{Z})$ s.t. $\Theta\chi = \xi = (\xi_1, \xi_1)$

Sketch of the proof I

Assume by contraddiction there exists $Sp(2,\mathbb{Z})$-invariant state ω (different from τ!)

$$\omega(W_\chi) = \begin{cases} 1 & \text{if } \chi = (0,0) \\ p^{(\chi)} \in \mathbb{C} & \text{else} \end{cases}$$

Goal: $Sp(2,\mathbb{Z})$-invariance $\iff p^{(\chi)} = 0$ for any $\chi \neq 0 \Longrightarrow \omega \equiv \tau$

(1) By positivity of $\omega \Longrightarrow \overline{\omega(W_\chi)} = \omega(W_\chi^*)$

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(7) In particular $Sp(2,\mathbb{Z})$-invariance $\Longrightarrow \omega(W_\chi) = \omega(W_\xi) = p$

Sketch of the proof II

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(8) Let $m, n \in \mathbb{N}$ s.t. $\frac{m}{n} \in \mathbb{N}$ and $n > 1$ and consider $\mathcal{V}_{\xi;\, m,n} \subset \mathcal{A}$

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(8) Let $m, n \in \mathbb{N}$ s.t. $\frac{m}{n} \in \mathbb{N}$ and $n > 1$ and consider $\mathcal{V}_{\xi;\, m,n} \subset \mathcal{A}$ with elements of the form

$$\mathfrak{a} = \alpha_0 W_{(0,0)} + \sum_{j \geq 1} \alpha_j W_{\Theta_j \xi} \qquad \text{with } \Theta_j := \begin{pmatrix} 1 + \frac{m}{n}(n-1)j & \frac{m}{n}j \\ n-1 & 1 \end{pmatrix} \in Sp(2, \mathbb{Z})$$

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(10) *Notation:* $\mathbf{H} = [\, p \,,\, 1 \,,\, q_m e^{\imath \varphi_{m,n}} \,,\, q_{2m} e^{\imath 2 \varphi_{m,n}} \,,\, q_{3m} e^{\imath 3 \varphi_{m,n}} \,,\, \dots]$

Sketch of the proof III

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$$\left(\text{idea: } \hbar := \frac{h}{2\pi} \text{ is irrational} \Rightarrow \exists m \in \mathbb{N} \text{ big enough s.t. } \frac{m}{d!} \in \mathbb{N} \text{ and } \left|(h\, m\, \xi_2^2) \mod (2\pi) - \frac{2\pi}{d}\right| < \frac{\varepsilon}{4d^2}\right)$$

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□

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`THANKS for your attention!`