A sufficient condition for the Bisognano-Wichmann property [1]

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[1] "*The Bisognano-Wichmann property on nets of standard subspaces, some sufficient conditions*", Annales Henri Poincaré (2017), https://doi.org/10.1007/s00023-017-0636-4, arXiv:1703.06831

1. Introduction: models, standard subspaces, one-particle nets

2. An algebraic condition for the B-W property on (generalized) one-particle nets

3. Counterexamples and remarks

1. Introduction

Introduction

In AQFT, $\mathbb{R}^{1+3}$ models are specified through Haag-Kastler axioms.
Let $\mathcal{H}$ be a fixed Hilbert space and $\mathbb{R}^{1+3} \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ be a map from the family of open causally closed regions in $\mathbb{R}^{1+3}$, to von Neumann algebras on $\mathcal{H}$ s.t. the following hold:

1. *Isotony*: if $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
2. *Locality*: if $O_1 \subset O_2'$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$
3. *Poincaré covariance and Positivity of the energy*: there exists a unitary, positive energy representation of the Poincaré group $\mathcal{P}_+^\uparrow$ acting covariantly on the net $\mathcal{A}$, namely

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \quad \forall g \in \mathcal{P}_+^\uparrow$$

4. *Existence and uniqueness of the vacuum*: there exists a unique (up to a phase) vector $\Omega \in \mathcal{H}$ s.t. $U(\mathcal{P}_+^\uparrow)\Omega = \Omega$
5. *Reeh-Schlieder*: $\mathcal{A}(O)\Omega$ is dense in $\mathcal{H}$

Introduction about models

We have two main characteristics in the model

- the algebraic structure $\mathcal{A} : O \mapsto \mathcal{A}(O)$
- the geometric structure $U : \mathcal{P}_+^\uparrow \to \mathcal{U}(\mathcal{H})$

Besides we recognize another character which is the vacuum state $\omega = \langle \Omega, \cdot \Omega \rangle$.

About the algebraic structure: **Tomita-Takesaki** theory.
Given an von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and a cyclic and separating vector $\Omega \in \mathcal{H}$ we can define the Tomita operator S_H, closure of the densely defined anti-linear involution:

$$\mathcal{H} \supset \mathcal{A}\Omega \ni a\Omega \longmapsto a^*\Omega \in \mathcal{A}\Omega \subset \mathcal{H}$$

and a polar decomposition $S_{\mathcal{A},\Omega} = J_{\mathcal{A},\Omega} \Delta^{1/2}_{A\,\Omega}$. $J_{\mathcal{A},\Omega}$ **modular conjugation** and $\Delta_{\mathcal{A},\Omega}$ **modular operator**. They satisfy

$$J_{\mathcal{A},\Omega} \Delta_{\mathcal{A},\Omega} J_{\mathcal{A},\Omega} = \Delta^{-1}_{\mathcal{A},\Omega}.$$

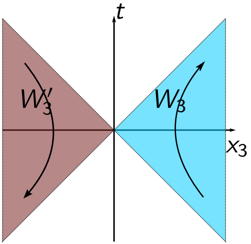
We have that

$$J_{\mathcal{A},\Omega} \mathcal{A} J_{\mathcal{A},\Omega} = \mathcal{A}' \quad \text{and} \quad \Delta^{it}_{\mathcal{A},\Omega} \mathcal{A} \Delta^{-it}_{\mathcal{A},\Omega} = \mathcal{A}$$

Introduction about the geometry

Let $W_\alpha = \{x \in \mathbb{R}^{1+3} : |x_0| < x_\alpha\}$ be a **wedge** in the direction x_α,
Λ_α be the pure Lorentz one-parameter group of **boosts** fixing W_α.
For instance we associate to W_3 the one parameter group of boosts

$$\Lambda_3(t)(p_0, p_1, p_2, p_3) = (\cosh(t)p_0 + \sinh(t)p_3, p_1, p_2, \sinh(t)p_0 + \cosh(t)p_3)$$



Sets of wedges: $\mathcal{W} = \mathcal{P}_+^\uparrow W_3$, $\mathcal{W}_0 = \mathcal{L}_+^\uparrow W_3$.
Λ_W: the **boosts** associated to $W \in \mathcal{W}$.

Introduction

Bisognano-Wichmann property [BW76]

$$U(\Lambda_W(2\pi t)) = \Delta^{-it}_{\mathcal{A}(W),\Omega}$$

Non-commutative structure + local algebras are type III

Modular covariance [BGL94]

$$\Delta^{-it}_{\mathcal{A}(W),\Omega}\mathcal{A}(O)\Delta^{it}_{\mathcal{A}(W),\Omega} = \mathcal{A}(\Lambda_W(2\pi t)O)$$

Given the algebraic structure and the vacuum state, the modular structure has a geometrical meaning [BGL95].

In particular modular covariance ensures the reconstruction of a unitary positive energy Poincaré representation + PCT opertor

Motivations

The Bisognano-Wichmann property is a **natural** requirement:

- Holds in Wightman fields [BW76]
- Gives a canonical structure to free fields [BGL02]
- Deduced by asymptoptic completeness [M01]
- Implies correct Spin-Statistics relation [GL95]
- Holds in conformal theories [GL96]
- Unnatural counterexample [Y94]
- Implies Essential Duality, i.e. $\mathcal{A}(W)' = \mathcal{A}(W')$ [BGL94]

Question: Can the B-W property be deduced by the axioms?

We propose an algebraic approach to the B-W property: we provide an algebraic sufficient condition on the covariant representation for the B-W property in the generalized one-particle - standard subspace - setting.

Standard Subspaces Araki, Brunetti, Eckmann, Guido, Longo, Osterwalder...

We start our discussion noticing that the modular theory of a von Neumann algebra is contained in its real structure.

A real linear closed subspace of an Hilbert space $H \subset \mathcal{H}$ is called **standard** if it is *cyclic* ($\overline{H + iH} = \mathcal{H}$) and *separating* ($H \cap iH = \{0\}$).
Symplectic complement: $H' = \{\xi \in \mathcal{H} : \Im\langle \xi, \eta \rangle = 0, \forall \eta \in H\}$

Analogue property to von Neumann algebras:

- $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\Omega \in \mathcal{H}$ is cyclic and separating iff $H_A = \overline{\mathcal{A}_{sa}\Omega}$ is cyclic and separating
- let $a\Omega \in H_A = \overline{\mathcal{A}_{sa}\Omega}$, $b\Omega \in H_{A'} = \overline{\mathcal{A}'_{sa}\Omega}$, then

$$\langle a\Omega, b\Omega \rangle = \langle b\Omega, a\Omega \rangle$$

thus $\Im\langle a\Omega, b\Omega \rangle = 0$

Standard Subspaces Araki, Brunetti, Echmann, Guido, Longo, Osterwalder...

Let H be a standard subspace. The associated Tomita operator is the closed anti-linear involution

$$S_H : H + iH \ni \xi + i\eta \longmapsto \xi - i\eta \in H + iH.$$

Its polar decomposition $S_H = J_H \Delta_H^{1/2}$ is s.t.

$$J_H \Delta_H J_H = \Delta_H^{-1}, \qquad \Delta_H^{it} H = H, \qquad J_H H = H'.$$

There is a **1-1 correspondence** $S_H \longleftrightarrow (J_H, \Delta_H) \longleftrightarrow H$.

Analogy with the Tomita-Takesaki theory.

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be v.N.a. with a cyclic and separating vector Ω and $H = \overline{A_{sa}\Omega}$. Then $S_{A,\Omega} = S_H$ coincide: let $a = a_1 + ia_2$ with $a_{1,2} \in \mathcal{A}_{sa}$

$$\begin{aligned} S_{\mathcal{A},\Omega} a\Omega &= S_{\mathcal{A},\Omega}(a_1 + ia_2)\Omega \\ &= (a_1 - ia_2)\Omega \\ &= S_H(a_1\Omega + ia_2\Omega) \end{aligned}$$

Subspaces **do not necessary** come from a von Neumann algebra.

Standard subspaces Poincaré covariant nets

A *U*-covariant net of standard subspaces $\mathcal{H}$ on the set $\mathcal{W}$ of wedge regions of the Minkowski spacetime is a map

$$H : \mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H}$$

that associates a closed real linear subspace $H(W)$ with each $W \in \mathcal{W}$, satisfying:

1. *Isotony*: if $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$;
2. *Locality*: For every wedge $W \in \mathcal{W}$ we have

$$H(W') \subset H(W)'$$

3. *Poincaré covariance and Positivity of the energy*:
   $U(g)H(W) = H(gW)$, $g \in \mathcal{P}_+^\uparrow$ and U has positive energy;
4. *Reeh-Schlieder property*: $H(W)$ is cyclic $\forall W \in \mathcal{W}$;

Nets satisfying 1.-4. will be denoted by (U, H)

5. *Bisognano-Wichmann property*:

$$\Delta_{H(W)}^{it} = U\big(\Lambda_W(-2\pi t)\big), \qquad \forall\ W \in \mathcal{W}\ ;$$

Standard subspaces Poincaré covariant nets

(at least) two reasons to study nets of standard subspaces:

1. they contain the modular structure of von Neumann algebras net
   $A(O) \mapsto H(O) = \overline{\mathcal{A}(O)_{sa}\Omega}$
2. they define one particle nets
   Scalar massive particle

$$H(O) = \overline{\{f \in \mathcal{C}^\infty(\mathbb{R}^{1+3}) : \operatorname{supp} f \subset O\}}$$

Scalar product:

$$\langle f, g \rangle = \int \overline{\hat{f}}(p)\hat{g}(p)\delta(p^2 - m^2)\theta(p_0)dp$$

Then second quantization gives the free field. **Not Canonical!**

Canonical one-particle net associated to a particle [BGL02]

U (anti-)unitary positive energy representation of $\mathcal{P}_+$
$\updownarrow$ 1-1
One particle nets satisfying **B-W property**

An algebraic condition for the B-W property

- We expect that under some conditions on the Poincaré representation, the canonical (generalized) one-particle net is "unique".

- One way to face this problem is to consider analytic extensions of wave functions (cf. Mund 2001 + Buchholz, Epstein 1985). There are some difficulties in extending the result to infinite multiplicity and direct integrals and to the massless case.

- We will provide an algebraic condition called **modularity condition** on a unitary p.e.r. of $\mathcal{P}_+^\uparrow$, sufficient to conclude B-W property on any standard subspace net the representation acts on.

2. A sufficient condition for the B-W property

An algebraic condition for the B-W property

Definition

A unitary, $\mathcal{P}_+^\uparrow$-p.e.r. U is **modular** if for any U-covariant net of standard subspaces H, namely any couple (U, H) the B-W property holds.

Definition

- $G_3^0 \doteq \{g \in \mathcal{L}_+^\uparrow : gW_3 = W_3\}$ the subgroup of $\mathcal{L}_+^\uparrow$ elements fixing W_3.
- $G_3 = \langle G_3^0, \mathcal{T} \rangle$, where $\mathcal{T}$ is the $\mathbb{R}^{1+3}$-translation group.
- For a general wedge $W \in \mathcal{W}$, G_W^0 and G_W are defined by the transitive action of $\mathcal{P}_+^\uparrow$ on wedges.

Definition

A unitary, positive energy $\mathcal{P}_+^\uparrow$-representation U satisfies the **modularity condition** if $r \in \mathcal{P}_+^\uparrow$ s.t. $rW = W'$

$$U(r) \in U(G_W)''. \qquad \text{(MC)}$$

A first remark

- It is sufficient to fix $W = W_3$ and $r = R_1(\pi)$, thus (MC) becomes

$$U(R_1(\pi)) \in U(G_3)''.$$

- For (almost) every $p = (p_0, p_1, p_2, p_3)$ in the forward light cone $\overline{V}^+ = \{p \in \mathbb{R}^{1+3} : p \cdot p \geq 0\}$

$$\begin{aligned} R_1(\pi)p =& (p_0, p_1, -p_2, -p_3) \\ =& \Lambda_3(t_p)(p_0, p_1, -p_2, p_3) \\ =& \Lambda_3(t_p)R_3(\theta_p)(p_0, p_1, p_2, p_3) \end{aligned} \tag{1}$$

  for a propery $t_p \in \mathbb{R}$ and $\theta_p \in [0, 2\pi]$.

- $G_3^0 = \langle \Lambda_3, R_3 \rangle$ and in particular $R_1(\pi)$ is an automorphism of a.e. orbits of G_3^0 on $\overline{V}^+$.

Modularity condition

Proposition

Let (U, H) be a Poincaré covariant net of standard subspaces. The strongly continuous map

$$Z_{H(W_3)} : \mathbb{R} \ni t \mapsto \Delta^{it}_{H(W_3)} U(\Lambda_3(2\pi t)).$$

is a one-parameter group and $Z_{H(W_3)}(t) \in U(G_3)'$.

Theorem

Let U be a unitary p.e.r. of the Poincaré group $\mathcal{P}_+^\uparrow$. If the condition (MC)

$$U(R_1(\pi)) \in U(G_3)''$$

holds on U, then any local U-covariant net of standard subspaces, satisfies the Bisognano-Wichmann property. In particular U is modular.

Idea of the proof: $Z_{H(W_3)}$ commutes with $U(R_1(\pi))$, then $Z_{H(W_3)} \equiv 1$ and B-W property holds.

The modularity condition

The condition (MC) can be extended easily to more general representations.

Proposition

Let U and $\{U_x\}_{x\in X}$ be unitary p.e.r. of $\mathcal{P}_+^\uparrow$ satisfying (MC)*. Let $\mathcal{K}$ be an Hilbert space, Let (X,μ) be a standard measure space.Then*

- (MC) *holds for $U \otimes 1_{\mathcal{K}} \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$.*
- *If $U_x|_{G_W}$ and $U_y|_{G_W}$ are disjoint for μ-a.e. $x \neq y$. Then $U = \int_X U_x d\mu(x)$ satisfies* (MC)*.*

Proposition

Assume that U satisfies (MC)*, then for every (U,H) the essential duality holds, namely $\mathcal{A}(W') = \mathcal{A}(W)'$.*

The modularity condition - the scalar case

The scalar representations have the following form

$$(U_{m,0}(a,g)\phi)(p) = e^{iap}\phi(g^{-1}p), \qquad (a,g) \in \mathbb{R}^{1+3} \rtimes \mathcal{L}_+^\uparrow = \mathcal{P}_+^\uparrow,$$

where

$$\phi \in \mathcal{H}_{m,0} \doteq L^2(\Omega_m, \delta(p^2 - m^2)\theta(p_0)d^4p),$$

and $\Omega_m = \{p = (p_0, p) \in \mathbb{R}^{1+3} : \; p^2 = p_0^2 - p^2 = m^2, p_0 \geq 0\}$, $m \geq 0$.

Proposition

Let U be a unitary, positive energy, irreducible scalar representation of the Poincaré group. Then U satisfies the modularity condition (MC) $U(R_1(\pi)) \in U(G_3)''$.

Proof uses that translation unitaries $\mathcal{T}$ generate MASA and G_3^0-orbits are $R_1(\pi)$-invariant

Theorem

Let $U = \int_{[0,+\infty)} U_m d\mu(m)$ where $\{U_m\}$ are (finite or infinite) multiples of the scalar representation of mass m, then U satisfies (MC). *In particular the B-W property hold for every (U, H) .*

3. Counterexamples and remarks

Counter-example

Counterexamples to modular covariance seem not so natural in Poincaré covariant framework (see for instance Yngvason 1994).

Counterexamples to B-W (with modular covariance). Let V be a $K-$real, bosonic, unitary representation of $\mathcal{L}_+^\uparrow$ on an Hilbert space $\mathcal{K} = K + iK$.
Let U_0 be the scalar, unitary irreducible representation of $\mathcal{P}_+^\uparrow$.

$$W \mapsto H_0(W) \in \mathcal{H}$$

the canonical BGL-net associated to U_0.
We can define the **new standard subspaces** net

$$\tilde{H} : W \longmapsto K \otimes H_0(W) \subset \tilde{\mathcal{H}} \doteq \mathcal{K} \otimes \mathcal{H}$$

There are **two representations** acting on $\tilde{H}$:

$$U_I : \widetilde{\mathcal{P}_+^\uparrow} \ni (a, A) \longmapsto 1_{\mathcal{K}} \otimes U_0(a, A) \in \mathcal{U}(\tilde{\mathcal{H}})$$

$$U_V : \widetilde{\mathcal{P}_+^\uparrow} \ni (a, A) \longmapsto V(A) \otimes U_0(a, A) \in \mathcal{U}(\tilde{\mathcal{H}})$$

Remarks

- Bisognano-Wichmann property holds for U_I (not for U_V).
- If U_0 is massive, U_V has infinitely many spins (possibly with finite multiplicity)
- if U_0 is massless, U_V is direct integral of infinite spin representations [LMR16]
- (MC) holds for scalar representations in $\mathbb{R}^{1+s}$, $s \geq 3$
- (MC) holds for irreducible finite helicity representations $\Rightarrow$ No one-particle nets associated (polarizations have to be combined)
- (MC) has to be generalized to include (at least) finite sum of spinorial representations
- Can (MC) be used to prove B-W for more general nets of von Neumann algebras?