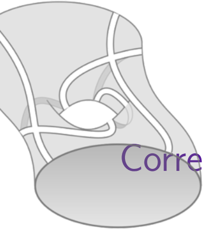

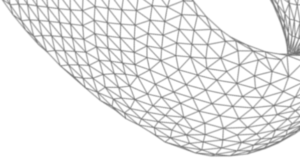

Correlation functions for colored tensor models

Schwinger-Dyson Equations

Carlos. I. Pérez-Sánchez

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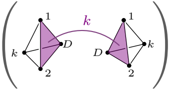
Mathematics Institute,
University of Münster

LQP 40, Max-Planck-Institut für Mathematik MIS
Leipzig, 23 June

Motivation

- Random Geometry framework ("Quantum Gravity")

$$\mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathcal{D}[g] \mathrm{e}^{-S_{\mathrm{EH}}[g]} \sim \sum\!\!\!\!\!\!\int_{(\text{diagram}) \in \substack{\text{topologies}\\ \text{geometries}}} \mu \left(\cdots \right)$$

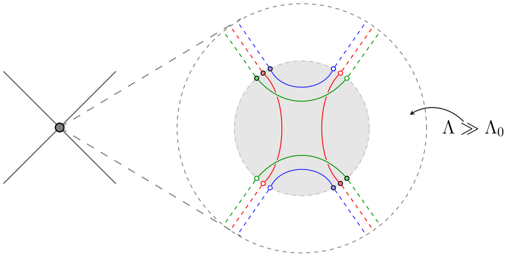


- Random matrices do that successfully for 2D. Random tensor models is a higher-dimensional arena, together with QFT-techniques, based on this idea
- Gurau-Witten model based on SYK-model (Sachdev-Ye–Kitaev). Random tensor methods useful in $\mathrm{AdS}_2/\mathrm{CFT}_1$ (Maldacena, Stanford)

Outline

- matrix and random tensor models
- Non-perturbative approach to quantum (coloured) tensor fields

φ^4-intearction Λ_0



  - graph-calculus: correlation functions
  - full Ward-Takahashi Identities: non-perturbative, systematic approach
  - Schwinger-Dyson equations: equations for the multiple-point functions (joint work with Raimar Wulkenhaar)

Random matrix theory: ensembles

- Nuclear physics (Wigner). Stochastics: $E \subset M_N(\mathbb{K})$:

$$\mathcal{Z} = \int_E \mathrm{d}\mu$$

Statistics of random eigenvalues; study limit $N \to \infty$; universality, μ-independence (tensor models too: book by R. Gurău)

- usually, for certain polynomial $P(x) = Nx^2/2 + N\,V(x)$,

$$\mathcal{Z} = \int_E \mathrm{d}M\,\mathrm{e}^{-\mathrm{Tr}\,P(M)} = \int_E \underbrace{\mathrm{d}M\,\mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\,M^2}}_{\mathrm{d}\mu_0}{}^{-N\mathrm{Tr}\,V(M)} = \int_E \mathrm{d}\mu_0\,\mathrm{e}^{-N\mathrm{Tr}\,V(M)}$$

- Kontsevich, Grosse-Wulkenhaar, Barrett-Glaser, ... models

- $V(M) = M^p$ ($p = 4, 6, 8$)

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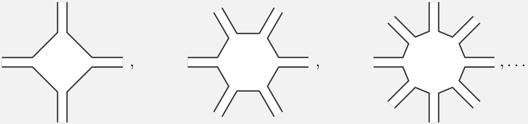
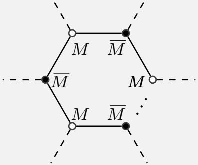
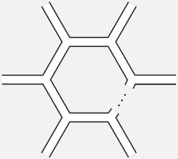
- $V(M) = M^p$ $(p = 4, 6, 8)$ 

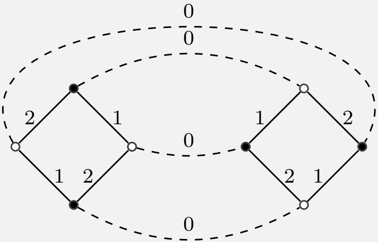
"Rank-2 tensor models"

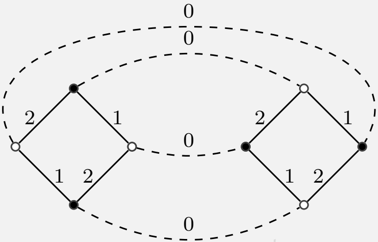
- For complex matrix models $\int \mathcal{D}[M,\overline{M}]\mathrm{e}^{-\mathrm{Tr}(MM^\dagger)-\lambda V(M,M^\dagger)}$



$=$ dual triangulation to  .

- For $V(M,\bar{M}) = \lambda \mathrm{Tr}((MM^\dagger)^2)$, different connected $\mathcal{O}(\lambda^2)$-graphs are





- rectangular matrices, $M \in \mathbb{M}_{N_1 \times N_2}(\mathbb{C})$ and $M \mapsto U^{(1)} M (U^{(2)})^{\mathrm{t}}$.
  $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$-invariants are $\mathrm{Tr}((MM^\dagger)^q)$, $q \in \mathbb{Z}_{\geq 1}$

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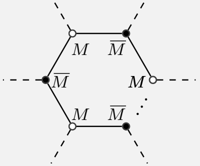
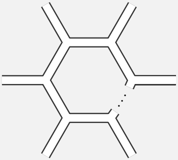


= dual triangulation to  .

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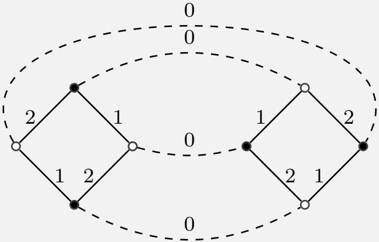

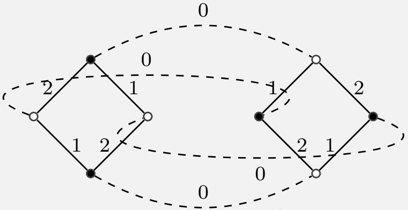

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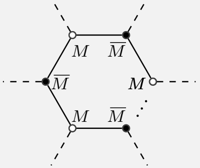
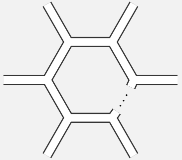


= dual triangulation to  .

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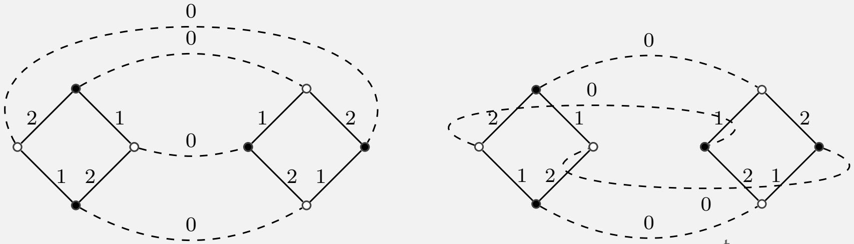

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Coloured Tensor Models

- a quantum field theory for tensors $\varphi_{a_1 \dots a_D}$ and $\overline{\varphi}_{a_1 \dots a_D}$
- the indices transform under *different* representations of

$$G = \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \dots \times \mathrm{U}(N_D)$$

- for $g \in G$, $g = (U^{(1)}, \dots, U^{(D)})$, $U^{(a)} \in \mathrm{U}(N_a)$,

$$\varphi_{a_1 a_2 \dots a_D} \overset{g}{\mapsto} (\varphi')_{a_1 a_2 \dots a_D} = U^{(1)}_{a_1 b_1} U^{(2)}_{a_2 b_2} \dots U^{(D)}_{a_D b_D} \varphi_{b_1 \dots b_D}$$

- the complex conjugate tensor $\overline{\varphi}_{a_1 a_2 \dots a_D}$ transforms as

$$\overline{\varphi}_{a_1 a_2 \dots a_D} \overset{g}{\mapsto} (\overline{\varphi}')_{a_1 a_2 \dots a_D} = \overline{U}^{(1)}_{a_1 b_1} \overline{U}^{(2)}_{a_2 b_2} \dots \overline{U}^{(D)}_{a_D b_D} \overline{\varphi}_{b_1 b_2 \dots b_D}$$

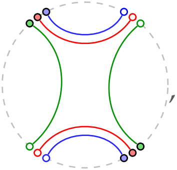
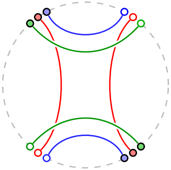
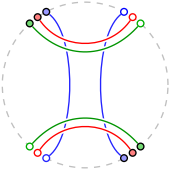
- G-invariants serve as *interaction vertices*

$$S[\varphi, \overline{\varphi}] = \sum_i \tau_i \mathrm{Tr}_{\mathcal{B}_i}(\varphi, \overline{\varphi}) = \mathrm{Tr}_{\mathcal{B}_2}(\overline{\varphi}, \varphi) + \sum_\alpha \lambda_\alpha \mathrm{Tr}_{\mathcal{B}_\alpha}(\overline{\varphi}, \varphi)$$

Feynman diagrams: Choose an action, for instance, the φ^4_3-theory,

$$S[\varphi,\bar{\varphi}] = \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi}) + \lambda\big(\mathrm{Tr}_{\mathcal{V}_1}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_2}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_3}(\varphi,\overline{\varphi})\big)$$

and

$\mathcal{V}_1 =$ , $\mathcal{V}_2 =$ , $\mathcal{V}_3 =$ ,

$$Z[J,\bar{J}] = \frac{\int \mathcal{D}[\varphi,\bar{\varphi}]\, \mathrm{e}^{\mathrm{Tr}_{\mathcal{B}_2}(\bar{J}\varphi)+\mathrm{Tr}_{\mathcal{B}_2}(\bar{\varphi}J)-N^2 S[\varphi,\overline{\varphi}]}}{\int \mathcal{D}[\varphi,\bar{\varphi}]\, \mathrm{e}^{-N^2 S[\varphi,\overline{\varphi}]}},$$ with $\mathrm{Tr}_{\mathcal{B}_2} \leftrightarrow$ 

$$\mathrm{d}\mu_C(\varphi,\overline{\varphi}) := \mathcal{D}[\varphi,\bar{\varphi}]\mathrm{e}^{-N^2 S_0[\varphi,\overline{\varphi}]} := \prod_a \frac{\mathrm{d}\varphi_a \mathrm{d}\overline{\varphi}_a}{2\pi\mathrm{i}} \mathrm{e}^{-N^2 \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi})}$$

- Write for Wick's contractions w.r.t. the Gaußian measure

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$ $\mathbf{p}$

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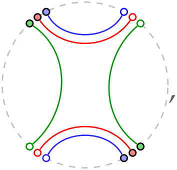
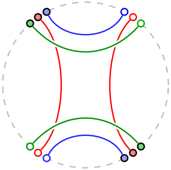
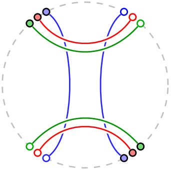
$\mathcal{V}_1 =$ , $\mathcal{V}_2 =$ , $\mathcal{V}_3 =$ ,

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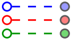
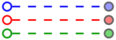
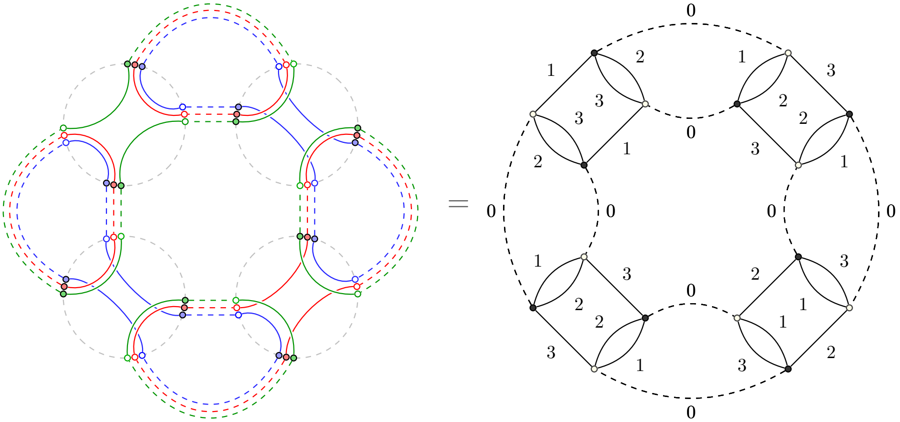
- Write  for Wick's contractions w.r.t. the Gaußian measure

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$  $\mathbf{p}$



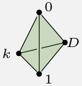
Vertex bipartite regularly edge–D-coloured graphs

- Feynman graphs of a model V, $\mathrm{Feyn}_D(V)$ are $(D+1)$-coloured. Crystallization theory or GEMs [Pezzana, '74] says all PL-manifolds of dimension D can be represented as $D+1$-coloured graphs, Grph_{D+1}.

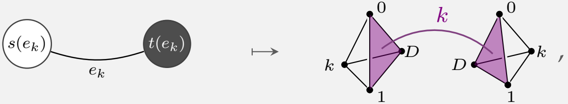
The complex $\Delta(\mathcal{G})$

- for each vertex $v \in \mathcal{G}^{(0)}$, add a D-simplex σ_v to $\Delta(\mathcal{G})$ with colour-labelled vertices $\{0, 1, \ldots, D\}$

$$\varphi, \bar{\varphi} \longmapsto v \longmapsto$$



- for each edge $e_k \in \mathcal{G}_k^{(1)}$ of arbitrary colour k, one identifies the two $(D-1)$-simplices $\sigma_{s(e_k)}$ and $\sigma_{t(e_k)}$ that do not contain the colour k.



edges come from either $\varphi_{a_1 \ldots a_k \ldots a_D} \delta_{a_k p_k} \bar{\varphi}_{p_1 \ldots p_k \ldots p_D} (k \neq 0)$ or $\varphi_{\mathbf{a}} \bar{\varphi}_{\mathbf{p}}$ $(k = 0)$.

[Gurău, '09] and [Bonzom, Gurău, Riello, Rivasseau, '11];

$$\mathcal{A}(\mathcal{G}) = \lambda^{V(\mathcal{G})/2} N^{\overbrace{F(\mathcal{G}) - \frac{D(D-1)}{4} V(\mathcal{G})}} = \exp(-S_{\text{Regge}}[N, D, \lambda])$$

$=: D - \frac{2}{(D-1)!}\omega(\mathcal{G})$ $\rightsquigarrow$ generalizes g; not topol. invariant

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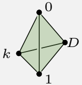

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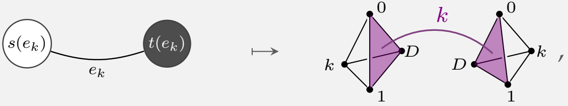

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- motivated by the WTI for matrix models by [Disertori-Gurau-Magnen-Rivasseau];
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$$\frac{\delta \log Z[J,\bar{J}]}{\delta (T_a^\alpha)_{m_a n_a}} = 0.$$

- this implies a relation of the type

$$\sum_{p_i \in \mathbb{Z}} E(m_a,n_a) \frac{\delta^2 Z[J,\bar{J}]}{\delta J_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} \bar{J}_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D}} = D_{J,\bar{J}} Z[J,\bar{J}]$$

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Expansion of the free energy

- $\operatorname{im}\partial_V = \partial\,\mathsf{Feyn}_D(V)$ is the *boundary sector* of the model V

$$W[J,\bar{J}] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar{\varphi}))\\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\; G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B})\,.$$

- Coloured automorphisms of $\mathcal{B}$

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar{J}_{\mathbf{y}^1}\cdots\bar{J}_{\mathbf{y}^k}$$

- Green's function $G^{(2k)}_{\mathcal{B}} = \partial^{2k}W[J,\bar{J}]/\partial\mathbb{J}(\mathcal{B})|_{J=\bar{J}=0}$
- $F: M_{D\times k(\mathcal{B})}(\mathbb{Z})\to\mathbb{C}; \quad \star:(F,\mathbb{J}(\mathcal{B}))\mapsto F\star\mathbb{J}(\mathcal{B}) = \sum_a F(a)\cdot\mathbb{J}(\mathcal{B})(a)$

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- Coloured automorphisms of $\mathcal{B}$

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar J_{\mathbf{y}^1}\cdots\bar J_{\mathbf{y}^k}$$

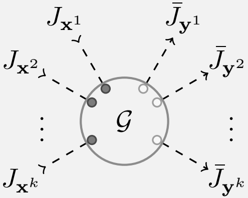
- Green's function $G^{(2k)}_{\mathcal{B}} = \partial^{2k}W[J,\bar J]/\partial\mathbb{J}(\mathcal{B})|_{J=\bar J=0}$
- $F: M_{D\times k(\mathcal{B})}(\mathbb{Z})\to\mathbb{C}$; $\quad\star:(F,\mathbb{J}(\mathcal{B}))\mapsto F\star\mathbb{J}(\mathcal{B}) = \sum_a F(a)\cdot\mathbb{J}(\mathcal{B})(a)$

- $\operatorname{im}\partial_V = \partial\,\mathsf{Feyn}_D(V)$ is the *boundary sector* of the model V

$$W[J,\bar J] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar\varphi))\\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\; G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B}).$$

- Coloured automorphisms of $\mathcal{B}$

$$\mathcal{B}_* : M_{D\times k(B)}(\mathbb{C}) \to M_{D\times k(B)}(\mathbb{C})$$

- 

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar J_{\mathbf{y}^1}\cdots\bar J_{\mathbf{y}^k}$$

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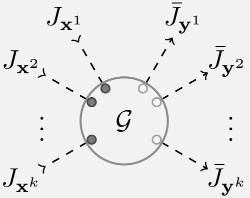
Expansion of the free energy

- $\operatorname{im}\partial_V = \partial\,\mathsf{Feyn}_D(V)$ is the *boundary sector* of the model V

$$W[J,\bar J] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar\varphi))\\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\, G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B})\,.$$

- Coloured automorphisms of $\mathcal{B}$

- 

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar J_{\mathbf{y}^1}\cdots\bar J_{\mathbf{y}^k}$$

- Green's function $G^{(2k)}_{\mathcal{B}} = \partial^{2k}W[J,\bar J]/\partial\mathbb{J}(\mathcal{B})|_{J=\bar J=0}$
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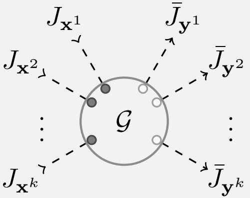
Expansion of the free energy

- $\operatorname{im}\partial_V = \partial\,\mathsf{Feyn}_D(V)$ is the *boundary sector* of the model V

$$W[J,\bar J] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar\varphi))\\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|} \, G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B}) \,.$$

- Coloured automorphisms of $\mathcal{B}$
- 

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar J_{\mathbf{y}^1}\cdots\bar J_{\mathbf{y}^k}$$
- Green's function $G^{(2k)}_{\mathcal{B}} = \partial^{2k} W[J,\bar J]/\partial\mathbb{J}(\mathcal{B})|_{J=\bar J=0}$
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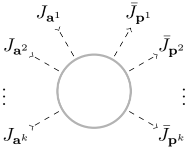
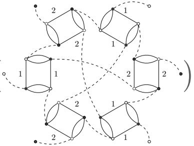
Green's functions

- $\mathcal{G} = $  $\rightsquigarrow \; \mathcal{B} = \partial\mathcal{G} \; \rightsquigarrow \; \mathbb{J}(\mathcal{B})\{\mathbf{a}^i\} = \prod_{i=1}^{k} J_{\mathbf{a}^i} \bar{J}_{\mathbf{p}^i}$

- One can derive a functional $X[J,\bar{J}]$ with respect to a graph. For instance:

$\partial\Big($  $\Big) =$ 

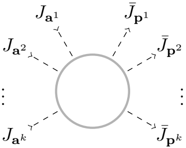
$$\frac{\partial X[J,\bar{J}]}{\partial \text{(graph with vertices } \mathbf{a},\mathbf{b},\mathbf{c})} = \frac{\partial^6 X[J,\bar{J}]}{\partial J_{\mathbf{a}}\, \partial J_{\mathbf{b}}\, \partial J_{\mathbf{c}}\, \partial \bar{J}_{a_1 c_2 b_3} \partial \bar{J}_{b_1 a_2 c_3} \partial \bar{J}_{c_1 b_2 a_3}}$$

- So: $\frac{\partial}{\partial \text{(graph } \mathbf{a},\mathbf{b},\mathbf{c})}\Big(\text{graph } \mathbf{e},\mathbf{f},\mathbf{g}\Big) = \delta^{\mathbf{e}}_{\mathbf{a}}\delta^{\mathbf{f}}_{\mathbf{b}}\delta^{\mathbf{g}}_{\mathbf{c}} + \delta^{\mathbf{g}}_{\mathbf{a}}\delta^{\mathbf{e}}_{\mathbf{b}}\delta^{\mathbf{f}}_{\mathbf{c}} + \delta^{\mathbf{f}}_{\mathbf{a}}\delta^{\mathbf{g}}_{\mathbf{b}}\delta^{\mathbf{e}}_{\mathbf{c}} \leftrightarrow \mathrm{Aut_c}(\text{graph}) \simeq \mathbb{Z}_3$

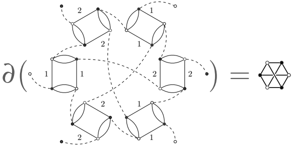
Lemma

$$G^{(2k)}_{\mathcal{B}}(\mathbf{a}^1,\ldots,\mathbf{a}^k) = \frac{\partial^{2k} W[J,\bar{J}]}{\partial \mathbb{J}(\mathcal{B})(\mathbf{a}^1,\ldots,\mathbf{a}^k)}\bigg|_{J=\bar{J}=0}$$ *are all non-trivial,* $\mathcal{B} \in \mathrm{im}\partial$

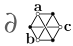
Green's functions

- $\mathcal{G} = \vdots$  $\vdots \;\rightsquigarrow\; \mathcal{B} = \partial\mathcal{G} \;\rightsquigarrow\; \mathbb{J}(\mathcal{B})\{\mathbf{a}^i\} = \prod_{i=1}^{k} J_{\mathbf{a}^i}\bar{J}_{\mathbf{p}^i}$

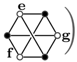
- One can derive a functional $X[J,\bar{J}]$ with respect to a graph. For instance:

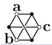


$$\frac{\partial X[J,\bar{J}]}{\partial\, \mathcal{B}_{\mathbf{a},\mathbf{b},\mathbf{c}}} = \frac{\partial^6 X[J,\bar{J}]}{\partial J_{\mathbf{a}}\,\partial J_{\mathbf{b}}\,\partial J_{\mathbf{c}}\,\partial \bar{J}_{a_1c_2b_3}\partial\bar{J}_{b_1a_2c_3}\partial\bar{J}_{c_1b_2a_3}}$$



- So: $\frac{\partial}{\partial\, \mathcal{B}_{\mathbf{a},\mathbf{b},\mathbf{c}}}\left(\mathcal{B}_{\mathbf{e},\mathbf{f},\mathbf{g}}\right) = \delta^{\mathbf{e}}_{\mathbf{a}}\delta^{\mathbf{f}}_{\mathbf{b}}\delta^{\mathbf{g}}_{\mathbf{c}} + \delta^{\mathbf{g}}_{\mathbf{a}}\delta^{\mathbf{e}}_{\mathbf{b}}\delta^{\mathbf{f}}_{\mathbf{c}} + \delta^{\mathbf{f}}_{\mathbf{a}}\delta^{\mathbf{g}}_{\mathbf{b}}\delta^{\mathbf{e}}_{\mathbf{c}} \leftrightarrow \mathrm{Aut}_{\mathrm{c}}(\mathcal{B}) \simeq \mathbb{Z}_3$



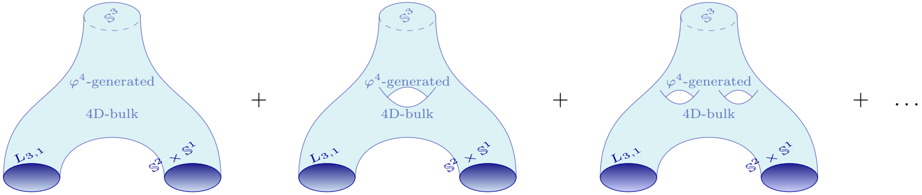


Lemma

$$G^{(2k)}_{\mathcal{B}}(\mathbf{a}^1,\ldots,\mathbf{a}^k) = \left.\frac{\partial^{2k} W[J,\bar{J}]}{\partial\mathbb{J}(\mathcal{B})(\mathbf{a}^1,\ldots,\mathbf{a}^k)}\right|_{J=\bar{J}=0}$$ *are all non-trivial*, $\mathcal{B} \in \mathrm{im}\partial$

Boundary graphs and bordisms interpretation

- since $\mathcal{B} = \partial\mathcal{G}$ represents the 'boundary of a simplicial complex that $\mathcal{G}$ triangulates', one can give a bordism-interpretation to the Green's functions
- for instance, if $|\Delta(\mathcal{B})| = \mathbb{S}^3 \sqcup (\mathbb{S}^2 \times \mathbb{S}^1) \sqcup L_{3,1} = \mathcal{M}$, then $G_{\mathcal{B}} = \partial W / \partial \mathcal{B}$ describes the bulk compatible with the triangulation of $\mathcal{M}$

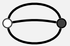


Lemma

The boundary sector of rank-D quartic melonic models is all of $\amalg\mathrm{Grph}_D$.

- For $D = 3$, all quartic vertices are melonic. The lowest order boundary connected graphs are:

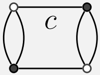
$\mathcal{M}$ — $k = 1$



ω: 0

$\mathrm{Aut}_c(\mathcal{M}) = \{*\}$ — #: 1

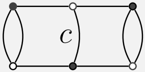
$\mathcal{V}_c$ — $k = 2$



ω: 0

$\mathrm{Aut}_c(\mathcal{V}_c) = \mathbb{Z}_2$ — #: 3

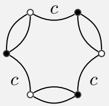
$\mathcal{E}_c$ — $k = 3$



ω: 0

$\mathrm{Aut}_c(\mathcal{E}_c) = \{*\}$ — #: 3

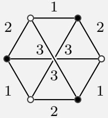
$\mathcal{Q}_c$ — $k = 3$



ω: 0

$\mathrm{Aut}_c(\mathcal{Q}_c) = \mathbb{Z}_3$ — #: 3

$K_c(3,3)$ — $k = 3$

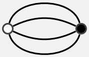


ω: 1

$\mathrm{Aut}_c(K_c(3,3)) = \mathbb{Z}_3$ — #: 1

- $D = 4$-connected boundary graphs

$\mathcal{M}$ — $k = 1$



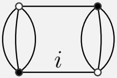
ω 0

$\mathrm{Aut}_c(\mathcal{M}) = \{*\}$

No counting needed

\# 1

$\mathcal{V}_i$ — $k = 2$



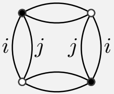
ω 0

$\mathrm{Aut}_c(\mathcal{V}_i) = \mathbb{Z}_2$

For any colour $i \in \{1,2,3,4\}$

\# 4

$\mathcal{N}_{ij}$ — $k = 2$



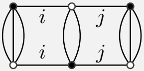
ω 1

$\mathrm{Aut}_c(\mathcal{N}_{ij}) = \mathbb{Z}_2$

Since $\mathcal{N}_{ij} = \mathcal{N}_{ji}$ one imposes $i < j$

\# 3

$\mathcal{E}_{ij}$ — $k = 3$



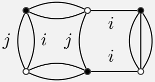
ω 0

$\mathrm{Aut}_c(\mathcal{E}_{ij}) = \{*\}$

$\mathcal{E}_{ij} = \mathcal{E}_{ji}$ $i < j$
$i, j \in \{1,2,3,4\}$

\# 6

$\mathcal{Q}_{ij}$ — $k = 3$



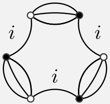
ω 1

$\mathrm{Aut}_c(\mathcal{Q}_{ij}) = \{*\}$

$\mathcal{Q}_{ij} \neq \mathcal{Q}_{ji}$
arbirary colours i, j

\# 12

$\mathcal{C}_i$ — $k = 3$

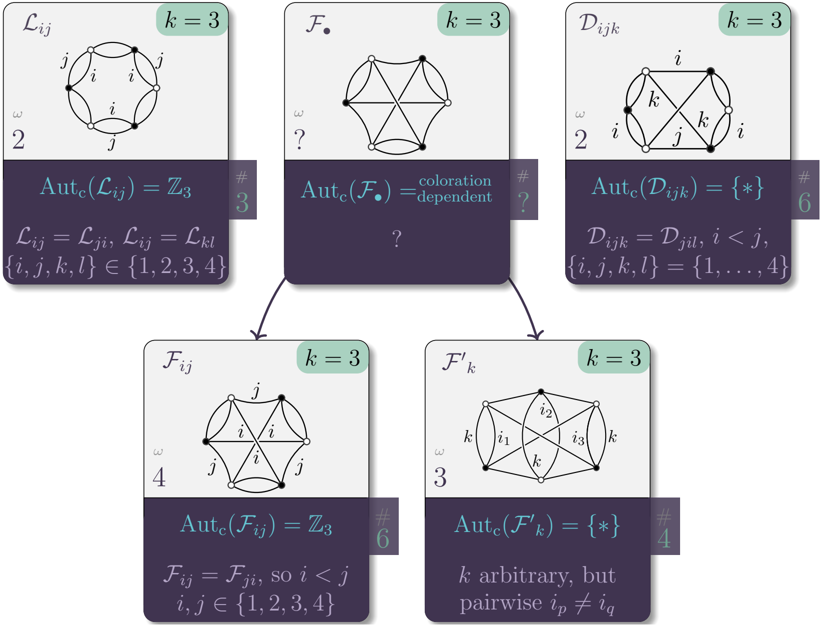


ω 0

$\mathrm{Aut}_c(\mathcal{C}_i) = \mathbb{Z}_3$

arbirary colour i

\# 4



$\mathcal{L}_{ij}$
$k=3$
ω
2
$\mathrm{Aut_c}(\mathcal{L}_{ij}) = \mathbb{Z}_3$
#
3
$\mathcal{L}_{ij} = \mathcal{L}_{ji}$, $\mathcal{L}_{ij} = \mathcal{L}_{kl}$
$\{i,j,k,l\} \in \{1,2,3,4\}$
$\mathcal{F}_\bullet$
$k=3$
ω
?
$\mathrm{Aut_c}(\mathcal{F}_\bullet) =$ coloration dependent
#
?
?
$\mathcal{D}_{ijk}$
$k=3$
ω
2
$\mathrm{Aut_c}(\mathcal{D}_{ijk}) = \{*\}$
#
6
$\mathcal{D}_{ijk} = \mathcal{D}_{jil}$, $i<j$,
$\{i,j,k,l\} = \{1,\dots,4\}$
$\mathcal{F}_{ij}$
$k=3$
ω
4
$\mathrm{Aut_c}(\mathcal{F}_{ij}) = \mathbb{Z}_3$
#
6
$\mathcal{F}_{ij} = \mathcal{F}_{ji}$, so $i<j$
$i,j \in \{1,2,3,4\}$
$\mathcal{F'}_k$
$k=3$
ω
3
$\mathrm{Aut_c}(\mathcal{F'}_k) = \{*\}$
#
4
k arbitrary, but
pairwise $i_p \neq i_q$

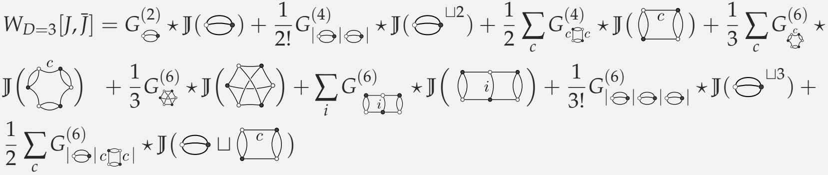
$W_{D=3}[J,\bar{J}]$

$W_{D=3}[J,\bar{J}] = G^{(2)}$  $\star\, \mathbb{J}($  $) +$

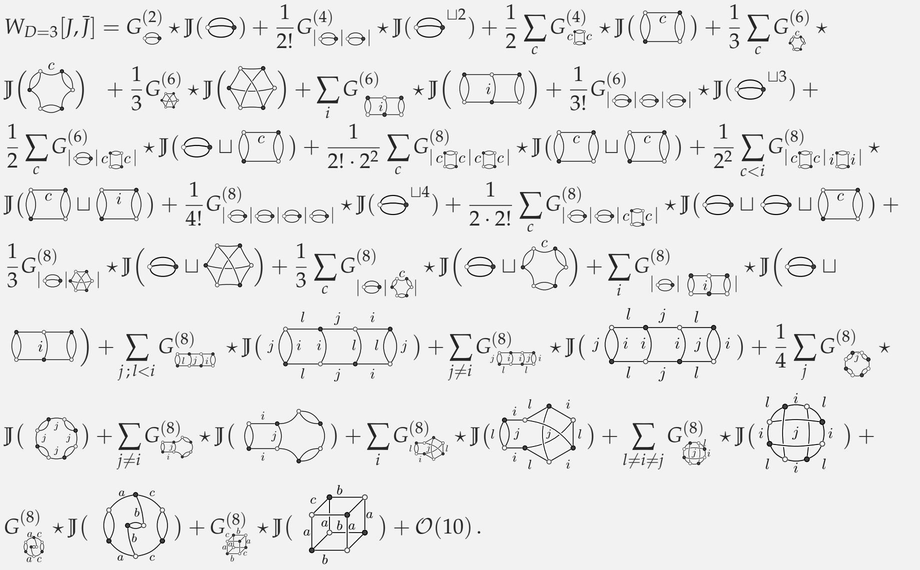
$W_{D=3}[J,\bar{J}]$

$$W_{D=3}[J,\bar{J}] = G^{(2)}_{\ominus} \star \mathbb{J}(\ominus) + \frac{1}{2!} G^{(4)}_{|\ominus|\ominus|} \star \mathbb{J}(\ominus^{\sqcup 2}) + \frac{1}{2}\sum_{c} G^{(4)}_{c\square c} \star \mathbb{J}(\square_c)$$

$W_{D=3}[J, \bar{J}]$



$W_{D=3}[J,\bar{J}]$



Theorem (Full Ward-Takahashi Identity for arbitrary tensor models)

If the kinetic form E in $\mathrm{Tr}_2(\bar\varphi, E\varphi)$ of a rank-D tensor model is such that

$$E_{p_1\ldots p_{a-1}m_a p_{a+1}\ldots p_D} - E_{p_1\ldots p_{a-1}n_a p_{a+1}\ldots p_D} = E(m_a, n_a) \quad \text{for each } a = 1,\ldots,D$$

then its partition function $Z[J,\bar J]$, as a consequence of unitary invariance of the measure $\delta Z[J,\bar J]/\delta(T^a)_{m_a n_a} = 0$, T^a a generator of $\mathfrak{u}(N)$), satisfies

$$\sum_{p_i\in\mathbb{Z}} \frac{\delta^2 Z[J,\bar J]}{\delta J_{p_1\ldots p_{a-1}m_a p_{a+1}\ldots p_D}\delta \bar J_{p_1\ldots p_{a-1}n_a p_{a+1}\ldots p_D}} - \Big(\delta_{m_a n_a} Y^{(a)}_{m_a}[J,\bar J]\Big)\cdot Z[J,\bar J]$$

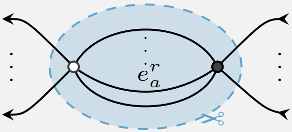
$$= \sum_{p_i\in\mathbb{Z}} \frac{1}{E(m_a,n_a)}\left(\bar J_{p_1\ldots m_a\ldots p_D}\frac{\delta}{\delta \bar J}_{p_1\ldots n_a\ldots p_D} - J_{p_1\ldots n_a\ldots p_D}\frac{\delta}{\delta J}_{p_1\ldots m_a\ldots p_D}\right) Z[J,\bar J]$$

where

$$Y^{(a)}_{m_a}[J,\bar J] := \sum_{k=1}^{\infty}\sideset{}{'}\sum_{\mathcal{B}\in \mathrm{im}\partial_V} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\langle\!\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B}\rangle\!\rangle_{m_a}$$

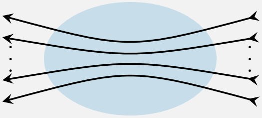
$$= \sum_{k=1}^{\infty}\sideset{}{'}\sum_{\mathcal{B}\in \mathrm{im}\partial_V} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|}\sum_{r=1}^{k}\Big(\Delta^{\mathcal{B}}_{m_a,r} G^{(2k)}_{\mathcal{B}}\Big)\star \mathbb{J}(\mathcal{B}\ominus e^r_a)\,.$$

Locally:



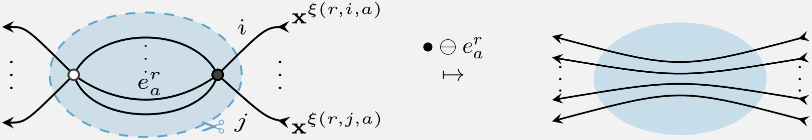
$\bullet \ominus e_a^r$

$\mapsto$



Defining $\mathcal{B} \mapsto \mathcal{B} \ominus e_a^r$ and $\Delta^{\mathcal{B}}_{m_a,r} : (\mathbb{C})^{\mathbb{Z}^{k\cdot D}} \to (\mathbb{C})^{\mathbb{Z}^{(k-1)\cdot D}}$ $(r = 1, \ldots, k)$

Locally:



Let $\mathbf{w} = (m_a, \{q_h\}_{h \in I(e^a_r)}, \{x_g^{\xi(r,g,a)}\}_{g \in A(e^a_r)})$ (colour-ordered);

- q_h is a dummy variable for each colour-h removed edge other than e_a^r
- $x_g^{\xi(r,g,a)}$ colour-g entry of $\mathbf{x}^{\xi(r,g,a)}$

Set, for $F : (\mathbb{Z}^D)^k \to \mathbb{C}$,

$$(\Delta^{\mathcal{B}}_{m_a,r} F)(\mathbf{x}^1, \ldots, \widehat{\mathbf{x}^r}, \ldots, \mathbf{x}^{k-1}) = \sum_{\{q_h\}} F(\mathbf{x}^1, \ldots, \mathbf{x}^{r-1}, \mathbf{w}(m_a, \mathbf{x}, \mathbf{q}), \ldots, \mathbf{x}^{k-1}) \text{ and}$$

$$\langle\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B} \rangle\rangle_{m_a} := \sum_{r=1}^{k} \left(\Delta^{\mathcal{B}}_{m_a,r} G^{(2k)}_{\mathcal{B}} \right) \star \mathbb{J}(\mathcal{B} \ominus e_a^r)$$

Examples of $\langle\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B} \rangle\rangle_{m_a}$

- for instance, for $D = 3$, $a = 2$

$$\left\langle\!\left\langle G^{(2)}_{\ominus}, \ominus \right\rangle\!\right\rangle_{m_2} = \Delta_{m_2,1} G^{(2)}_{\ominus} \star \mathbb{J}(\emptyset) = \sum_{q_1, q_3 \in \mathbb{Z}} G^{(2)}_{\ominus}(q_1, m_2, q_3)\,.$$

- In $D = 4$, for $\mathcal{F}'_c =$, one has

$$\mathcal{F}'_c \ominus e^1_a = \mathcal{F}'_c \ominus e^3_a = \qquad , \quad \mathcal{F}'_c \ominus e^2_a =$$

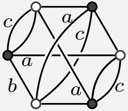
$$\begin{aligned} \left\langle\!\left\langle G^{(6)}_{\mathcal{F}'_c}, \mathcal{F}'_c \right\rangle\!\right\rangle_{m_a} = \ & \Delta_{m_a,1} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad) \\ & + \Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad) \\ & + \Delta_{m_a,3} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}(\qquad) \end{aligned}$$

$$\begin{aligned} & \Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, \mathbf{z}) \\ = & \sum_{q_c} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, (m_a, y_b, q_c, z_d), \mathbf{z}) \end{aligned}$$

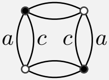
Examples of $\langle\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B}\rangle\rangle_{m_a}$

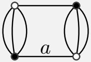
- for instance, for $D=3$, $a=2$

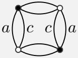
$\langle\langle G^{(2)}$  ,  $\rangle\rangle_{m_2} = \Delta_{m_2,1} G^{(2)}$  $\star \mathbb{J}(\emptyset) = \sum_{q_1,q_3\in\mathbb{Z}} G^{(2)}$  $(q_1, m_2, q_3)\,.$

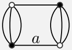
- In $D=4$, for $\mathcal{F}'_c =$ 

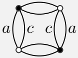
, one has

$\mathcal{F}'_c \ominus e^1_a = \mathcal{F}'_c \ominus e^3_a =$ 

, $\mathcal{F}'_c \ominus e^2_a =$ 

$\langle\langle G^{(6)}_{\mathcal{F}'_c}, \mathcal{F}'_c \rangle\rangle_{m_a} = \Delta_{m_a,1} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

$+ \Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

$+ \Delta_{m_a,3} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

$$\Delta_{m_a,2} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y},\mathbf{z}) = \sum_{q_c} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, (m_a, y_b, q_c, z_d), \mathbf{z})$$

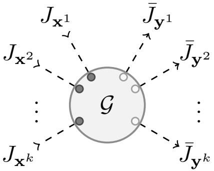
Graph-generated functionals

- For any $k \in \mathbb{N}$ let

$$\mathcal{F}_{D,k} := \{(\mathbf{y}^1, \ldots, \mathbf{y}^k) \in M_{D\times k}(\mathbb{Z}) \,|\, y_c^\alpha \neq y_c^\nu \text{ for all } c = 1, \ldots, D, \\ \text{for all } \alpha, \nu = 1, \ldots, k, \alpha \neq \nu\}\,.$$

- We define the graph derivative of a functional $X[J,\bar{J}]$ with respect to $\mathcal{B}$ at $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^k) \in \mathcal{F}_{D,k}$ as

$$\frac{\partial X[J,\bar{J}]}{\partial \mathcal{B}(\mathbf{X})} := \frac{\delta^{2k(\mathcal{B})} X[J,\bar{J}]}{\delta(\mathbb{J}(\mathcal{B}))(\mathbf{X})}\bigg|_{\substack{J=0\\ \bar{J}=0}} = \prod_{\alpha=1}^{k} \frac{\delta}{\delta J_{\mathbf{x}^\alpha}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}} X[J,\bar{J}]\bigg|_{\substack{J=0\\ \bar{J}=0}}$$



- $\Upsilon_{s_a}^{(a)}[J,\bar{J}] = \sum_{\mathcal{C}\in\Omega_V} \mathfrak{f}_{\mathcal{C},s_a}^{(a)} \star \mathbb{J}(\mathcal{C})$. The derivative w.r.t. connected $\mathcal{B} \in \Omega_V$ is

$$\frac{\partial \Upsilon_{s_a}^{(a)}[J,\bar{J}]}{\partial\, \mathcal{B}(\mathbf{X})} = \sum_{\hat{\sigma}\in \mathrm{Aut_c}(\mathcal{B})} (\sigma^* \mathfrak{f}_{\mathcal{B}})(\mathbf{X}),$$

where $(\sigma^* \mathfrak{f}_{\mathcal{B}})(\mathbf{x}^1, \ldots, \mathbf{x}^{k(\mathcal{B})}) := \mathfrak{f}_{\mathcal{B}}(\mathbf{x}^{\sigma^{-1}(1)}, \ldots, \mathbf{x}^{\sigma^{-1}(k(\mathcal{B}))})\,.$

Schwinger-Dyson equations

SDEs for the $\varphi^4_{\text{mel},D}$-model ($k \geq 2$) [C.P., Raimar Wulkenhaar]

Let $D \geq 3$ and let $\mathcal{B}$ be a connected boundary graph of the quartic melonic model, $\mathcal{B} \in$ Feyn$_D(\varphi^4_{\text{m},D}) =$ Grph$^{\text{II,cl}}_D$. Let $\mathbf{s} = \mathbf{y}^1$, where $\mathcal{B}_*(\mathbf{X}) = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$ for any $\mathbf{X} \in \mathcal{F}_{k(\mathcal{B}),D}$. The $(2k)$-point Schwinger-Dyson equation corresponding to $\mathcal{B}$ is

$$\left(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat{a}}} \quad (s_a, \mathbf{q}_{\hat{a}})\right) G^{(2k)}_{\mathcal{B}}(\mathbf{X}) \qquad (1)$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \left\{ \sum_{\hat{\sigma} \in \text{Aut}_c(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B},s_a}(\mathbf{X}) + \sum_{\rho > 1} \frac{1}{E(y^\rho_a, s_a)} Z_0^{-1} \frac{\partial Z[J,\bar{J}]}{\partial \varsigma_a(\mathcal{B}; 1, \rho)(\mathbf{X})} \right.$$

$$\left. - \sum_{b_a} \frac{1}{E(s_a, b_a)} \left[G^{(2k)}_{\mathcal{B}}(\mathbf{X}) - G^{(2k)}_{\mathcal{B}}(\mathbf{X}|_{s_a \to b_a}) \right] \right\}$$

for all $\mathbf{X} \in \mathcal{F}_{D,k(\mathcal{B})}$. Here $s_a \mathbf{q}_a = (q_1, q_2, \ldots, q_{a-1}, s_a, q_{a+1}, \ldots, q_D)$.

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$$\Big(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat{a}}} \qquad (s_a, \mathbf{q}_{\hat{a}})\Big) G_{\mathcal{B}}^{(2k)}(\mathbf{X}) \tag{1}$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \Bigg\{ \sum_{\hat{\sigma} \in \mathrm{Aut}_c(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B},s_a}(\mathbf{X}) + \sum_{\rho>1} \frac{1}{E(y_a^\rho, s_a)} Z_0^{-1} \frac{\partial Z[J,\bar{J}]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})}$$

$$- \sum_{b_a} \frac{1}{E(s_a, b_a)} \big[G_{\mathcal{B}}^{(2k)}(\mathbf{X}) - G_{\mathcal{B}}^{(2k)}(\mathbf{X}|_{s_a \to b_a}) \big] \Bigg\}$$

for all $\mathbf{X} \in \mathcal{F}_{D,k(\mathcal{B})}$. Here $s_a \mathbf{q}_a = (q_1, q_2, \ldots, q_{a-1}, s_a, q_{a+1}, \ldots, q_D)$.

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$$\left(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat{a}}} G^{(2)}(s_a, \mathbf{q}_{\hat{a}})\right) G^{(2k)}_{\mathcal{B}}(\mathbf{X}) \tag{1}$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \left\{ \sum_{\hat{\sigma} \in \text{Aut}_{\text{c}}(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B},s_a}(\mathbf{X}) + \sum_{\rho > 1} \frac{1}{E(y^\rho_a, s_a)} Z_0^{-1} \frac{\partial Z[J,\bar{J}]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})} \right.$$

$$\left. - \sum_{b_a} \frac{1}{E(s_a, b_a)} \left[G^{(2k)}_{\mathcal{B}}(\mathbf{X}) - G^{(2k)}_{\mathcal{B}}(\mathbf{X}|_{s_a \to b_a}) \right] \right\}$$

for all $\mathbf{X} \in \mathcal{F}_{D,k(\mathcal{B})}$. Here $s_a \mathbf{q}_a = (q_1, q_2, \ldots, q_{a-1}, s_a, q_{a+1}, \ldots, q_D)$.

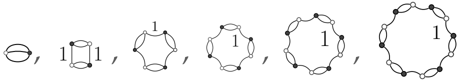
A simple quartic model

- Proposal of a model with $V[\varphi,\bar\varphi] = \lambda \cdot$ 

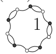
$$S_0[\varphi,\bar\varphi] = \mathrm{Tr}_2(\bar\varphi, E\varphi) = \sum_{\mathbf{x}\in\mathbb{Z}^3} \bar\varphi_{\mathbf{x}}(m^2+|\mathbf{x}|^2)\varphi_{\mathbf{x}}\,, \quad |\mathbf{x}|^2 = x_1^2+x_2^2+x_3^2,$$

- The boundary sector is determined by:

$\partial\,\mathrm{Feyn}_3($  $) = \{\mathcal{B} \in \mathrm{Grph}_3^{\mathrm{II}} : \mathcal{B}$ has connected components in $\Theta\}$

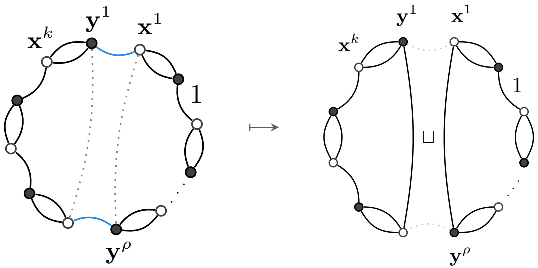
being $\Theta = \{$  $, \ldots\}$.

- Let $\mathcal{X}_{2k}$ be the graph in Θ with $2k$ vertices, and $G^{(2k)} = G^{(2k)}_{\mathcal{X}_{2k}}$:

$G^{(2)} = G^{(2)}$ , $G^{(4)} = G^{(4)}$ , $G^{(6)} = G^{(6)}$ , $G^{(8)} = G^{(8)}$ , $G^{(10)} = G^{(10)}$ .

Preparing the SDEs

- $\mathrm{Aut_c}(\mathcal{X}_{2k}) = \mathbb{Z}_k$



so $\varsigma_1(\mathcal{X}_{2k}; 1, \rho) = \mathcal{X}_{2\rho-2} \sqcup \mathcal{X}_{2k-2\rho+2}$

- $Y^{(1)}_{s_1}[J,\bar{J}] = \sum_{k=0}^{\infty} \mathfrak{f}_{2k,s_1} \star \mathbb{J}(\mathcal{X}_{2k}) + \sum_{\mathcal{C}\ \text{disconnected}} \mathfrak{f}^{(1)}_{\mathcal{C},s_1} \star \mathbb{J}(\mathcal{C})$

$$\mathfrak{f}_{2,s_1} = \frac{1}{2} \sum_{r=1}^{2} \left(\Delta_{s_1,r} G^{(4)}_{|\ominus|\ominus|} + \Delta_{s_1,r} G^{(4)} \right)$$

$$\mathfrak{f}_{2k,s_1} = \frac{1}{k} \Delta_{s_1,1} G^{(2k+2)}_{|\ominus|\mathcal{X}_{2k}|} + \frac{1}{k+1} \sum_{r=1}^{k} \Delta_{s_1,r} G^{(2k+2)}, \text{ for } k \geq 2\,.$$

Schwinger-Dyson equations ($\mathbb{S}^3$-geometries)

Let $\mathcal{B}$ be a connected boundary graph of the quartic model with $2k$ vertices ($k \geq 1$), $\mathcal{B} \in \mathrm{Feyn}_3($ 1  1 $)$. Let $\mathbf{s} = \mathbf{y}^1 = (x_1^1, x_2^r, x_3^r)$, where

$$(\mathcal{X}_{2k})_*(\mathbf{X}) = (\mathbf{y}^1, \ldots, \mathbf{y}^k), \qquad \mathbf{X} \in \mathcal{F}_{3,k}\,.$$

The $(2k)$-point Schwinger-Dyson equation corresponding to $\mathcal{B}$ is

$$\left(1 + \frac{2\lambda}{m^2 + |\mathbf{s}|^2} \sum_{q,p\,\in\mathbb{Z}} G^{(2)}(s_1, q, p)\right) \cdot G^{(2k)}(\mathbf{X})$$

$$= \frac{2\lambda}{m^2 + |\mathbf{s}|^2} \left\{ \frac{\delta_{1,k}}{2\lambda} - \sum_{\hat{\sigma}\in\mathbb{Z}_k} \sigma^* \mathfrak{f}_{2k,s_1}(\mathbf{X}) - \sum_{\rho>1} \frac{Z_0^{-1}}{[(y_1^\rho)^2 - s_1^2]} \cdot \frac{\partial Z[J,\bar{J}]}{\partial \varsigma_1(\mathcal{X}_{2k}; 1, \rho)(\mathbf{X})} \right.$$

$$\left. + \sum_{q\in\mathbb{Z}} \frac{1}{s_1^2 - q^2} \left[G^{(2k)}(\mathbf{X}) - G^{(2k)}(\mathbf{X}|_{s_1 \to q}) \right] \right\}.$$

The exact 2-point equation for the 11-model is given, for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$, by

$$\left(1 + \frac{2\lambda}{m^2 + |\mathbf{x}|^2} \sum_{q,p \in \mathbb{Z}} G^{(2)}(x_1, q, p)\right) \cdot G^{(2)}(\mathbf{x}) \tag{2}$$

$$= \frac{1}{m^2 + |\mathbf{x}|^2} + \frac{(-2\lambda)}{m^2 + |\mathbf{x}|^2} \left\{ \sum_{p,q \in \mathbb{Z}} G^{(4)}_{|\ominus|\ominus|}(x_1, q, p, \mathbf{x}) + G^{(4)}(\mathbf{x}, \mathbf{x}) \right.$$

$$\left. - \sum_{q \in \mathbb{Z}} \frac{1}{x_1^2 - q^2} \left[G^{(2)}(x_1, x_2, x_3) - G^{(2)}(q, x_2, x_3) \right] \right\}.$$

whose melonic ('planar') limit is (conjecturally, expected to be)

$$\left(m^2 + |\mathbf{x}|^2 + 2\lambda \sum_{q,p \in \mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1, q, p)\right) \cdot G^{(2)}_{\mathrm{mel}}(\mathbf{x}) \tag{3}$$

$$= 1 + 2\lambda \sum_{q \in \mathbb{Z}} \frac{1}{x_1^2 - q^2} \left[G^{(2)}_{\mathrm{mel}}(x_1, x_2, x_3) - G^{(2)}_{\mathrm{mel}}(q, x_2, x_3) \right].$$

The higher multipoint functions satisfy:

$$
\begin{aligned}
&\left(1+\frac{2\lambda}{m^2+|\mathbf{s}|^2}\sum_{q,p\in\mathbb{Z}} G^{(2)}(x_1^1,q,p)\right)\cdot G^{(2k)}(\mathbf{x}^1,\ldots,\mathbf{x}^k) \\
&= \frac{(-2\lambda)}{m^2+|\mathbf{s}|^2}\Bigg\{ \sum_{l=1}^{k}\Bigg[\frac{1}{k}\sum_{p,q\in\mathbb{Z}} G^{(2k+2)}_{|\ominus|\mathcal{X}_{2k}|}(x_1^1,q,p;\mathbf{x}^{1+l},\ldots,\mathbf{x}^{k+l}) \qquad (4)\\
&\qquad + \frac{1}{k+1}\sum_{r=1}^{k} G^{(2k+2)}(\mathbf{x}^{1+l},\mathbf{x}^{2+l},\ldots,\mathbf{x}^{r+l-1},x_1^1,x_2^{r+l-1},x_2^{r+l-1},\mathbf{x}^{r+l},\ldots,\mathbf{x}^{k+l})\Bigg] \\
&\qquad + \sum_{\rho=2}^{k}\frac{1}{[(x_1^\rho)^2-(x_1^1)^2]}\Big(G^{(2\rho-2)}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1})\cdot G^{(2k-2\rho+2)}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k)\Big) \\
&\qquad - \sum_{q\in\mathbb{Z}}\frac{G^{(2k)}(x_1^1,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)-G^{(2k)}(q,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)}{(x_1^1)^2-q^2}\Bigg\}.
\end{aligned}
$$

The exact $2k$-equation (melonic limit, conjecturally)

$$\left(1+\frac{2\lambda}{m^2+|\mathbf{s}|^2}\sum_{q,p\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1^1,q,p)\right)\cdot G^{(2k)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^k) \tag{5}$$

$$= \frac{(-2\lambda)}{m^2+|\mathbf{s}|^2}\left[\sum_{\rho=2}^{k}\frac{1}{(x_1^\rho)^2-(x_1^1)^2}\left(G^{(2\rho-2)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1})\cdot G^{(2k-2\rho+2)}_{\mathrm{mel}}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k)\right)\right.$$

$$\left. -\sum_{q\in\mathbb{Z}}\frac{G^{(2k)}_{\mathrm{mel}}(x_1^1,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)-G^{(2k)}_{\mathrm{mel}}(q,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)}{(x_1^1)^2-q^2}\right].$$

Conclusions & outlook

- (Coloured) tensor field theories [Ben Geloun, Bonzom, Carrozza, Gurău, Krajewski, Oriti, Ousmane-Samary, Rivasseau, Ryan, Tanasa, Vignes-Tourneret,...] provide a framework for $3 \leq D$-dimensional random geometry
  - A bordism interpretation of the correlation functions was given
  - A new Ward-Takahashi identity [C.P.] (bare parameters) based that for matrix models has been found
    - non-perturbative
    - universal: same for each interaction vertices
    - full (information has been recovered)
    - provides a method to systematically obtain exact equations for correlation functions
  - It has been used to derive the full tower of SDE [C.P.-Wulkenhaar]
- Outlook: Apply these techniques for SYK-like [Sachdev-Ye-Kitaev] models [Witten]

References

- M. Disertori, R. Gurău, J. Magnen, and V. Rivasseau.
  Phys. Lett., B649:95–102, 2007.
  arXiv:hep-th/0612251.
- H. Grosse and R. Wulkenhaar
  Commun. Math. Phys. **329** (2014) 1069
  arXiv:1205.0465 [math-ph].
- R. Gurău,
  Commun. Math. Phys. **304**, 69 (2011)
  arXiv:0907.2582 [hep-th] .
- C. I. Pérez-Sánchez, R. Wulkenhaar
  arXiv:1706.07358
- C. I. Pérez-Sánchez.
  arXiv:1608.08134 and arXiv:1608.00246
- E. Witten
  arXiv:1610.09758v2 [hep-th].

Thank you for your attention!