Constructive Matrix Theory for Higher Order Interaction

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Motivation

Matrices are everywhere in physics, randomness is also everywhere.
Random matrices are applicable to:

- the description of the energy spectra of heavy nuclei
- the large N_c limit of QCD
- random surfaces, 2d quantum gravity
- transport in disordered systems
- string theory
- number theory
- biology
- ....

The constructive program for matrix models was limited to the case of quartic interaction (V. Rivasseau, 2007; R. Gurau, T. Krajewski 2014).

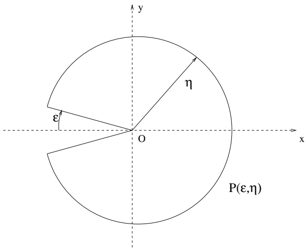
The Model

We study the models with monomial interactions of arbitrarily high even order:

$$Z(\lambda, N) := \int dMdM^\dagger \, e^{-NS(M,M^\dagger)}$$

$$S(M, M^\dagger) := \mathrm{Tr}\{MM^\dagger + \lambda(MM^\dagger)^p\} \, ,$$

where M is a complex matrix $N \times N$, $p \geq 2$ is integer, λ is complex.
The main result: the free energy is analytic for λ in an open "pacman domain", $P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\}$



Loop Vertex Representation (Expansion)

To proof the main result we apply and develop LVR(E) machinery, which is in contrast with traditional constructive methods is *not* based on cluster expansions nor involves small/large field conditions.

- Like Feynman's perturbative expansion, the LVR(E) allows to compute connected quantities at a glance: log(forests) = *trees*.
- Typically, the convergence of the LVR(E) implies Borel summability of the usual perturbation series.
- The LVR(E) is an *explicit repacking* of infinitely many subsets of pieces of Feynman amplitudes.
- In the case of the matrix and tensor models with a non-trivial $N \to \infty$ limit, the Borel summability obtained by LVR(E) (or just analyticity) is *uniform* in the size N of the model.

Main steps of LVR(E)

1. The divergence of the standard perturbation theory is caused by the to singular growth of the interaction potential at large fields. Therefore, we derive effective action $S_{eff}(M)$, providing polynomial interaction ====> Log-type interaction.
2. Taylor expansion
$$e^{S_{eff}(M)} = \sum_{n=0}^{\infty} \frac{(S_{eff}(M))^n}{n!} .$$
3. Replication of fields, by introducing degenerate Gaussian measure, so
$$(S_{eff}(M))^n \ ====> \ \prod_{i}^{n} S_{eff}(M_i) .$$
4. Application of the BKAR forest formula.
5. Taking the log by reducing the sum over forests to the sum over trees.
6. Derivation of the bounds for tree LVR(E) amplitudes.

Effective action (1)

The partition function is

$$
\begin{aligned}
Z(\lambda, N) &:= \int d\widetilde{M} d\widetilde{M}^\dagger \, e^{-NS(\widetilde{M},\widetilde{M}^\dagger)} \, , \\
S(\widetilde{M}, \widetilde{M}^\dagger) &:= \mathrm{Tr}\{\widetilde{M}\widetilde{M}^\dagger + \lambda(\widetilde{M}\widetilde{M}^\dagger)^p\} \, .
\end{aligned}
$$

To derive effective action, we perform a change of variables

$$MM^\dagger = \widetilde{M}\widetilde{M}^\dagger + \lambda(\widetilde{M}\widetilde{M}^\dagger)^p \, .$$

Then,

$$\widetilde{M}\widetilde{M}^\dagger = MM^\dagger \, T_p(-\lambda(MM^\dagger)^{p-1}) \, ,$$

where T_p is a solution of the Fuss-Catalan algebraic equation

$$zT_p^p(z) - T_p(z) + 1 = 0 \, .$$

Effective action (2)

We define

$$X := MM^{\dagger}\,, \quad A(X) := XT_p(-\lambda X^{p-1})\,.$$

The Jacobian is

$$J = \left|\frac{\delta A(X)}{\delta X}\right| = \left|\frac{A(X)\otimes \mathbf{1} - \mathbf{1}\otimes A(X)}{X\otimes \mathbf{1} - \mathbf{1}\otimes X}\right|$$

and the effective action is

$$S_{eff}(M, M^{\dagger}) = \log J = \log\left[\mathbf{1}_{\otimes} + \lambda \sum_{k=0}^{p-1} A^{k}(X)\otimes A^{p-1-k}(X)\right].$$

Holomorphic calculus

In the following forest/tree expansion we need to compute multiple derivatives ∂_M, $\partial_{M^\dagger}$, therefore we need to simplify the effective action.

Given a holomorphic function f on a domain containing the spectrum of a square matrix X, Cauchy's integral formula yields a convenient expression for $f(X)$,

$$f(X) = \oint_\Gamma dw \frac{f(w)}{w - X},$$

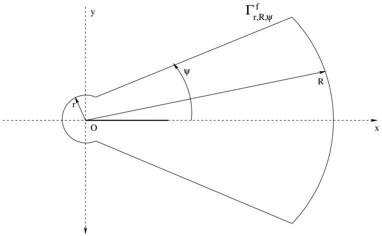
provided the contour Γ encloses the full spectrum of X.

Holomorphic calculus

We can therefore write

$$A(X) = \oint_\Gamma du\; a(\lambda, u)\, \frac{1}{u - X}$$

where $a(\lambda, z) = zT_p(-\lambda z^{p-1})$ and the contour Γ is a *finite* keyhole contour enclosing all the spectrum of X.



The matrix derivative can be easily obtained as

$$\frac{\partial A}{\partial X} = \oint_\Gamma du\; a(\lambda, u) \frac{1}{u - X} \otimes \frac{1}{u - X}.$$

Effective action (3)

The effective action is now given by

$$S_{eff}(\lambda, X) = \int_0^\lambda dt \oint_{\Gamma_1} dv_1 \oint_{\Gamma_2} dv_2 \Big\{ \oint_{\Gamma_0} du\ \phi(t, u, v_1, v_2) \psi(t, v_1, v_2) \Big\} \mathcal{R}(v_1, v_2, X)$$

where

$$\phi(\lambda, u, v_1, v_2) = -\sum_{k=1}^{p-2} \frac{1}{v_1 - u} \frac{1}{v_2 - u} a(\lambda, u) \partial_\lambda \left[\lambda a^k(\lambda, v_1) a^{p-k-1}(\lambda, v_2) \right],$$

$$\psi(\lambda, v_1, v_2) = -\frac{2}{v_1 - v_2} a(\lambda, v_1) \partial_\lambda \left[\lambda a^{p-1}(\lambda, v_2) \right],$$

$$\mathcal{R}(v_1, v_2, X) = \left[\mathrm{Tr} \frac{1}{v_1 - X} \right] \left[\mathrm{Tr} \frac{1}{v_2 - X} \right].$$

How to compute $\log Z$

The effective action provides a way to generate convergent expansion for the partition function

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dM dM^{\dagger} \exp\{-N \mathrm{Tr} X\} S_{eff}^{n} .$$

To compute the logarithm we apply the forest/tree expansion:

forests ====> log ====> trees

Theorem (Brydges-Kennedy-Abdesselam-Rivasseau). *Let $\phi : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{C}$ be a smooth, sufficiently derivable function. Then:*

$$\phi(1, \ldots, 1) = \sum_{F \text{ forest}} \int_0^1 \prod_{(i,j)\in F} du_{ij} \left(\frac{\partial^{|E(F)|}\phi}{\prod_{(i,j)\in F} \partial x_{ij}} \right) (v_{ij}^F) ,$$

where v_{ij}^F is given by:

$$v_{ij}^F = \begin{cases} \inf_{(k,l)\in P_{i\leftrightarrow j}^F} u_{kl} & \text{if } P_{i\leftrightarrow j}^F \text{ exists} \\ 0 & \text{if } P_{i\leftrightarrow j}^F \text{ does not exist} \end{cases} ,$$

and $|E(F)|$ is the number of edges in the forest F.

BKAR forest formula

$n = 2$

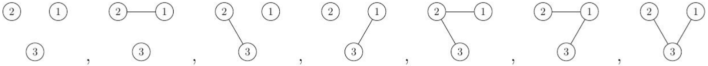
$$\phi(1) = \phi(0) + \int_0^1 dt_{12} \Big(\frac{\partial \phi}{\partial x_{12}} \Big)(t_{12})$$

The first term corresponds to the empty forest ($|E(F)| = 0$) and the second one to the full forest ($|E(F)| = 1$).

BKAR forest formula

$n = 3$

$$\begin{aligned}\phi(1,1,1) = \phi(0,0,0) &+ \int_{[0,1]} dt_{12} \left(\frac{\partial \phi}{\partial x_{12}}\right)(t_{12},0,0) \\ &+ \int_{[0,1]} dt_{23} \left(\frac{\partial \phi}{\partial x_{23}}\right)(0,t_{23},0) + \int_{[0,1]} dt_{13} \left(\frac{\partial \phi}{\partial x_{13}}\right)(0,0,t_{13}) \\ &+ \int_{[0,1]^2} dt_{12}dt_{23} \left(\frac{\partial^2 \phi}{\partial x_{12}\partial x_{23}}\right)(t_{12},t_{23},\inf(t_{12},t_{23})) \\ &+ \int_{[0,1]^2} dt_{12}dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{12}\partial x_{13}}\right)(t_{12},\inf(t_{12},t_{13}),t_{13}) \\ &+ \int_{[0,1]^2} dt_{23}dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{23}\partial x_{13}}\right)(\inf(t_{23},t_{13}),t_{23},t_{13}) \ .\end{aligned}$$



Preparing the application of the forest formula

To generate a convergent LVE, we start by expanding $\exp[S_{eff}(X)]$

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dM dM^{\dagger} \exp\{-N \mathrm{Tr} X\} \, S_{eff}^{n} \, .$$

The next step (replicas) is to replace (for the order n) the integral over the single $N \times N$ complex matrix M by an integral over an n-tuple of such $N \times N$ matrices $M_i, 1 \leq i \leq n$.

$$d\mu =====> d\mu_C$$

with a degenerate covariance $C_{ij} = N^{-1} \;\; \forall i, j$.

$$\int d\mu_C M^{\dagger}_{i|ab} M_{j|cd} = C_{ij} \delta_{ad} \delta_{bc} ,$$

$M_{i|ab}$ is the matrix element in the row a and column b of the matrix M_i.

$$d\mu_C \Leftrightarrow d\mu \delta(M_1 - M_2) \cdots \delta(M_{n-1} - M_n)$$

Preparing the application of the forest formula

Now the partition function is

$$Z(\lambda, N) \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu_C \prod_{i=1}^{n} S_{eff}(M_i) \,,$$

it can be represented as a sum over the set $\mathfrak{F}_n$ of forests $\mathcal{F}$ on n labeled vertices by applying the BKAR formula.

For this, we replace the covariance $C_{ij} = N^{-1}$ by $C_{ij}(x) = N^{-1} x_{ij}$ ($x_{ij} = x_{ji}$) evaluated at $x_{ij} = 1$ for $i \neq j$ and $C_{ii}(x) = N^{-1}\ \forall i$.

Then the Taylor BKAR formula yields

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}\in\mathfrak{F}_n} \int dw_{\mathcal{F}}\, \partial_{\mathcal{F}} \mathcal{Z}_n \Big|_{x_{ij}=x_{ij}^{\mathcal{F}}(w)}$$

where

$$\int dw_{\mathcal{F}} := \prod_{(i,j)\in\mathcal{F}} \int_0^1 dw_{ij}\,, \quad \partial_{\mathcal{F}} := \prod_{(i,j)\in\mathcal{F}} \frac{\partial}{\partial x_{ij}}\,,$$

$$\mathcal{Z}_n := \int d\mu_{C(x)} \prod_{i=1}^{n} S_{eff}(M_i)$$

$$x_{ij}^{\mathcal{F}} = \begin{cases} \inf_{(k,l)\in P_{i\leftrightarrow j}^{\mathcal{F}}} w_{kl} & \text{if } P_{i\leftrightarrow j}^{\mathcal{F}} \text{ exists}\,, \\ 0 & \text{if } P_{i\leftrightarrow j}^{\mathcal{F}} \text{ does not exist}\,. \end{cases}$$

In this formula w_{ij} is the weakening parameter of the edge (i,j) of the forest, and $P_{i\leftrightarrow j}^{\mathcal{F}}$ is the unique path in $\mathcal{F}$ joining i and j when it exists.

The derivative with respect to x_{ij} transforms into derivatives with respect to M_i and $M_j^\dagger$:

$$\partial_{\mathcal{F}} \;\; ====> \;\; \partial_{\mathcal{F}}^M = \prod_{(i,j)\in\mathcal{F}} \mathrm{Tr}\left[\frac{\partial}{\partial M_i^\dagger}\frac{\partial}{\partial M_j}\right].$$

$$\partial_M \mathrm{Tr}\frac{1}{v-X} \;=\; \mathrm{Tr}\left[\frac{1}{v-X}\otimes M^\dagger \frac{1}{v-X}\right]$$

$$\begin{aligned}\partial_M \partial_{M^\dagger} \mathrm{Tr}\frac{1}{v-X} &= \mathrm{Tr}\left[\frac{1}{v-X}M \otimes \frac{1}{v-X}\otimes M^\dagger \frac{1}{v-X}\right] \\ &+ \mathrm{Tr}\left[\frac{1}{v-X}\otimes M^\dagger \frac{1}{v-X}M \otimes \frac{1}{v-X}\right] \\ &+ \mathrm{Tr}\left[\frac{1}{v-X}\otimes \mathbf{1} \otimes \frac{1}{v-X}\right].\end{aligned}$$

The latter derivatives connect "loop vertices".

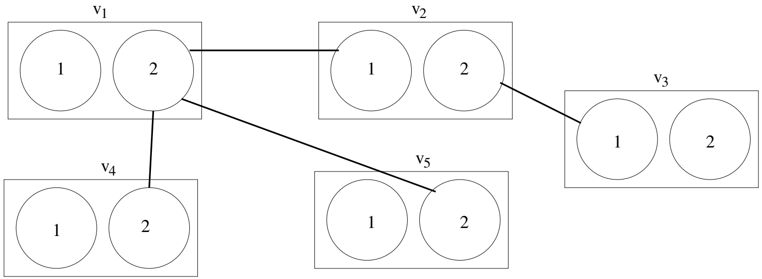


Figure: A tree of $n-1$ lines on n loop vertices (depicted as rectangular boxes, hence here $n=5$) defines a forest of $n+1$ connected components or cycles $\mathcal{C}$ on the $2n$ elementary loops, since each vertex contains exactly two loops. To each such cycle corresponds a trace of a given product of operators in the LVE.

Bounds

- Factorization of the traces provides the possibility to use the trace bound
$$|\mathrm{Tr}[O...O]| \leq \|O\|...\|O\| \,.$$
- On the keyhole contours, the derivatives of the matrix part of the effective action are bounded by
$$\begin{aligned} \|\frac{1}{v_j^i - X^i}\| &\leq K(1+|v_j^i|)^{-1}, \\ \|\frac{1}{v_j^i - X^i} M^i\| &\leq K(1+|v_j^i|)^{-1/2} \ldots \end{aligned}$$
and for the scalar part we have:
$$\begin{aligned} |T_p(z)| &\leq \frac{K}{(1+|z|)^{1/p}}, \\ |\frac{d}{dz} T_p(z)| &\leq \frac{K}{(1+|z|)^{1+\frac{1}{p}}}. \end{aligned}$$

Bounds

- For each tree amplitude, uniformly in N

$$|A_{\mathcal{T}}(\lambda, N)| \leq K^n |\lambda|^{\kappa_p n}$$

- The number of trees grows just as $n!$,
- what is compensated by the symmetry factor $\frac{1}{n!}$.

The main theorem

Theorem

For any $\epsilon > 0$ *there exists* η *small enough such that the LVR(E) expansion is absolutely convergent and defines an analytic function of* λ*, uniformly bounded in* N*, in the "pacman domain"*

$$P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\},$$

a domain which is uniform in N*. Here absolutely convergent and uniformly bounded in* N *means that for fixed* ϵ *and* η *as above there exists a constant* K *independent of* N *such that for* $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \leq K < \infty.$$

Conclusions and Outlook

- Similar results are derived for the Hermitian matrix models
- The techniques used in the work are based on the re-parametrization invariance and highlight its importance.
- The utilization of the holomorphic calculus methods drastically simplifies the construction.
- The latter simplification provides the chance to look from a new perspective to the QFT models, tensor models...