Lattice-gauge theory and Duflo-Weyl quantization

Alexander Stottmeister

joint work with Arnaud Brothier

University of Rome "Tor Vergata"
Department of Mathemtics

Göttingen
February 2, 2018

Unterstützt von / Supported by

Alexander von Humboldt
Stiftung / Foundation

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5. Construction of states and type classification of algebras

Time-dependent Born-Oppenheimer approximation

How to extract QFTs on curved backgrounds from quantum gravity?

Problems

- Mathematical framework?
  - $\rightarrow$ beyond $\hbar \longrightarrow 0+$
  - $\rightarrow$ Hamiltonian approach
- Approximate dynamics?
  - $\rightarrow$ systematics beyonds $O(t)$
  - $\rightarrow$ no fibered Hamiltonians

$$H_\varepsilon = \int^{\oplus} d\xi \; H_0(\xi) + f(-i\varepsilon\nabla) \otimes 1$$

Main idea

Utilize/adapt space-adiabatic perturbation theory [Panati, Spohn,Teufel; 2003].

Basic ingredient

Suitable pseudo-differential calculus ($\rightarrow$ Equivariant Duflo-Weyl quantization).

Time-dependent Born-Oppenheimer approximation

Space-adiabatic perturbation theory

Wish list

(1) Coupled quantum dynamical system $(\mathcal{H}, (\hat{H}, D(\hat{H})))$

(2) Splitting of the dynamics (controlled by parameter ε)

$$\mathcal{H} = \mathcal{H}_{\text{slow}} \otimes \mathcal{H}_{\text{fast}}$$

(3) ε-dependent deformation (de)quantization

$$\widehat{\cdot}^{\,\varepsilon} : \overbrace{S^{\infty}}^{\text{symbols}}(\varepsilon, \underbrace{\Gamma}_{\text{slow phase space}}, \mathcal{B}(\mathcal{H}_{\text{fast}})) \subset C^{\infty}(\Gamma, \mathcal{B}(\mathcal{H}_{\text{fast}})) \longrightarrow \mathcal{L}(\mathcal{H})$$

(4) Asymptotic expansion of Hamiltonian symbol (up to smoothing operators $S^{-\infty}$)

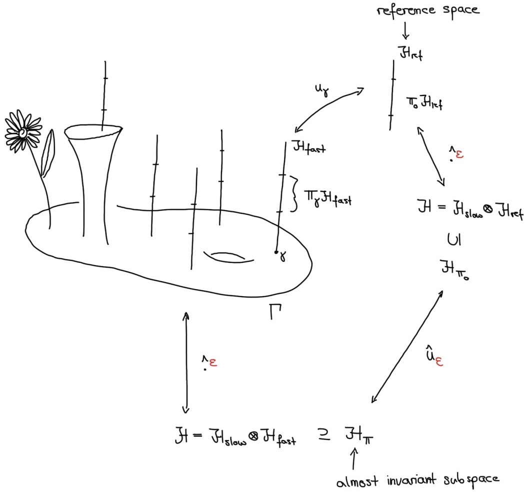
$$H_{\varepsilon} \sim \sum_{k=0}^{\infty} \varepsilon^{k} H_{k}, \;\; H_{k} \in S^{\rho-k}$$

$$\hat{H} = \widehat{H_{\varepsilon}}^{\varepsilon}$$

(5) Conditions on the (point-wise) spectrum $\sigma_*(H_0) = \{\sigma(H_0(\gamma))\}_{\gamma \in \Gamma}$

Time-dependent Born-Oppenheimer approximation

Space-adiabatic perturbation theory



Upshot

Construct effective dynamics in $\mathcal{H}_{\pi_0}$ (ε-independent subspace).

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Operator-algebraic approaches to lattice-gauge theory

Hamiltonian formulation [Kogut, Susskind; 1975]

Operator-algebraic formulations

- Mathematical framework
  - → fixed finite lattices [Kijowski, Rudolph; 2002]
  - → fixed infinite lattice [Grundling, Rudolph; 2013]
  - → inductive limit over finite lattices [Arici, Stienstra, van Suijlekom; 2017]
- Common aspect
  - → Replace the classical edge phase space T^*G by the C^*-algebra $C(G) \rtimes G$ (G-version of CCR).

Problem

$C(G) \rtimes G$ is not unital. This complicates constructions.

Observation

Equivariant Duflo-Weyl quantization is related to $C(G) \rtimes G$ as well. It requires a unital extension to be well-defined.

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Unitary representations of Thompson's groups

Reconstruction of CFTs from subfactors [Jones; 2014]

1+1 dimensional chiral CFTs

- $\{\mathcal{A}(I)\}_{I\subset S^1}$ (conformal net of type III factors)
- $\mathcal{A}(I)\subset\mathcal{B}(I)$, extensions give subfactors
  $\rightarrow$ Characterized by algebraic data (planar algebras).

Main idea [Jones; 2014]

Use planar-algebra data to reconstruct CFTs from subfactors.

$\rightarrow$ Define a functor from binary planar forest to Hilbert spaces.

$$\underbrace{\mathsf{Y}}_{\text{basic forest}} \longmapsto \underbrace{(\mathcal{H}_1\rightarrow\mathcal{H}_2)}_{\text{“spin doubling”}}$$

$\rightarrow$ Gives discrete CFT models (Thompson group symmetry).

Observation

These discrete CFT models fit into the same framework as those defined by equivariant Duflo-Weyl quantization.

Functor $\longleftrightarrow$ Inductive limit over lattices/graphs

The basic construction

The elementary phase space

Some ingredients

- Γ will be modeled on T^*G.
- Pseudo-differential calculus for T^*G?
  - $\rightarrow$ Start from a strict deformation quantization [Rieffel; 1990], [Landsman; 1993].
- $T^*G \cong G \times \mathfrak{g}$, $\mathfrak{g} = \mathsf{Lie}(G)$, $n = \dim(G)$.
- $\exp : \mathfrak{g} \longrightarrow G$ is onto and locally one-to-one ($U \rightarrow V$).
  - $\rightarrow$ Use $\exp$ to relate the Haar measure on G and the Lebesgue measure on $\mathfrak{g}$:

$$\int_{V \subset G} dg\ f(g) = \int_{U \subset \mathfrak{g}} dX\ j(X)^2\ f(\exp(X)),\ f \in C_c^\infty(V)$$

$$j(H) = \prod_{\alpha \in R_+} \frac{\sin(\alpha(H)/2)}{\alpha(H)/2},\ H \in \mathfrak{t}\ \text{(restriction to a maximal torus)}$$

The basic construction

Operators from convolution kernels

Fibre-wise Fourier transform

- For $X_h = \exp^{-1}(h)$, $\sigma \in C^\infty_{\mathrm{PW},U_\varepsilon}(\mathfrak{g})\hat{\otimes}C^\infty(G)$, $U_\varepsilon = \varepsilon^{-1}U$, define

$$F^\varepsilon_\sigma(h,g) = \check{\sigma}^1_\varepsilon(X_h,g) = \int_{\mathfrak{g}^*} \frac{d\theta}{(2\pi\varepsilon)^n} e^{\frac{i}{\varepsilon}\theta(X_h)}\sigma(\theta,g),\ \varepsilon \in (0,1].$$

$\rightarrow$ $F^\varepsilon_\sigma \in C^\infty(G)\hat{\otimes}C^\infty(G)$ gives the kernel of a Kohn-Nirenberg-type ΨDO.

- Deform the construction to obtain a Duflo-Weyl-type ΨDO:
  - $\rightarrow$ Locally: $F^{W,\varepsilon}_\sigma(h,g) = F^\varepsilon_\sigma(h, \sqrt{h^{-1}}g)$.
  - $\rightarrow$ Globally: Use the wrapping map Φ^{DW} [Dooley, Wildberger; 1993]

$$< \Phi^{DW}(\check{\sigma}^1_\varepsilon)(g), f >_G =< \check{\sigma}^1_\varepsilon(\exp(-\tfrac{1}{2}(\ .\ ))g), j \cdot \exp^* f >_{\mathfrak{g}},\ f \in C^\infty(G)$$

Duflo-Weyl formula for $C^\infty(T^*G) \longrightarrow \mathcal{L}(L^2(G))$

Operators are obtained from the integrated left-regular representation:

$$(Q^{DW}_\varepsilon(\sigma)f) =< \Phi^{DW}(\check{\sigma}^1_\varepsilon)(g), \iota^* R^*_g f >_G,\ f \in C^\infty(G)$$

The basic construction

Properties of the quantization

Theorem (generalization of [Landsman; 1993])

$$Q_\varepsilon^{DW} : C^\infty_{\mathrm{PW},U}(\mathfrak{g}) \hat{\otimes} C^\infty(G) \longrightarrow \mathcal{K}(L^2(G)) \cong C(G) \rtimes G$$

is a non-degenerate strict deformation quantization on $(0,1]$ w.r.t. to the canonical Poisson structure on T^*G. Furthermore, the G-CCR are satisfied:

$$Q_\varepsilon^{DW}(\{\sigma_f, \sigma_{f'}\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_f), Q_\varepsilon^{DW}(\sigma_{f'})] = 0,$$

$$Q_\varepsilon^{DW}(\{\sigma_X, \sigma_f\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_X), Q_\varepsilon^{DW}(\sigma_f)] = R_X f,$$

$$Q_\varepsilon^{DW}(\{\sigma_X, \sigma_Y\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_X), Q_\varepsilon^{DW}(\sigma_Y)] = i\varepsilon R_{[X,Y]},$$

for $\sigma_f(\theta, g) = f(g)$, $f \in C^\infty(G)$, and $\sigma_X(\theta, g) = \theta(X)$, $X \in \mathfrak{g}$ (momentum map of the Hamiltonian G-action).

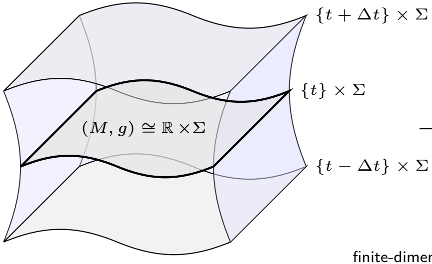
Pseudo-differential calculus

The quantization Q_ε^{DW} allows for a pseudo-differential calculus on T^*G.

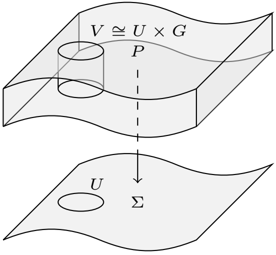
→ Symbol spaces, asymptotic completeness, star product, etc.

→ Complications due to the compactness of G.

A projective phase space for lattice-gauge theories



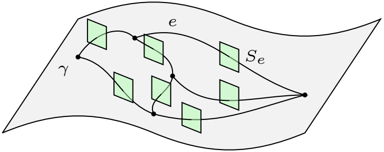
initial data formulation
temporal gauge



finite-dimensional
projections

Hamiltonian formulation:
- $M \cong \mathbb{R} \times \Sigma$ - Cauchy foliation

Hamiltonian gauge-field formulation:
- Σ - Cauchy surface
- G - structure group (compact)
- A, E - gauge field, conjugate electric field
- $D_A E = 0$ - Gauß constraint



Basic functionals:
- $g_e(A)$ - Holonomy
- $P^e_X(A, E; S_e)$ - Flux

Phase space:

$$\Gamma \subset \overline{\Gamma} = \varprojlim_\gamma \Gamma_\gamma$$
$$\Gamma_\gamma = T^* G^{|E(\gamma)|}$$

A projective phase space for lattice-gauge theories

Structure of the finite-dimensional phase spaces

The induced Poisson structure

Using a suitable regularization of the infinite-dimensional Poisson structure, the basic functionals w.r.t. a given graph γ generate the G-CCR of $T^*G^{|E(\gamma)|}$:

$$\begin{aligned} \{f(g_e), f'(g_{e'})\}_\gamma(A,E) &= 0, \\ \{P^e_X, f'(g_{e'})\}_\gamma(A,E) &= \delta^{e,e'}(R_X f')(g_{e'}(A)), \\ \{P^e_X, P^{e'}_Y\}_\gamma(A,E) &= -\delta_{e,e'} P^e_{[X,Y]}(A,E) \end{aligned}$$

Operations on graphs

The basic functionals behave naturally w.r.t. operations on graphs:

$e = e_2 \circ e_1 : g_e(A) = g_{e_2}(A) g_{e_1}(A),$ (composition)

$e \mapsto e^{-1} : g_{e^{-1}}(A) = g_e(A)^{-1},\ P^{e^{-1}}_X(A,E) = -P^e_{Ad_{g_e(A)}(X)}(A,E),$ (inversion)

$e \mapsto \emptyset$: drop dependence. (removal)

Composition for fluxes

The behavior of fluxes w.r.t. composition is more complicated:

A projective phase space for lattice-gauge theories

Some inductive constructions

Action of the gauge group

The gauge group $\mathcal{G}$ has a natural action on the finite-dimensional phase spaces.

$\rightarrow$ Gauge transformations act at the vertices of the graphs.

$\rightarrow$ The action on $\mathcal{L}(C^\infty(\Gamma_\gamma))$ is induced by the action on convolution kernels:

$$\alpha_\gamma(\{g_v\}_{v\in V(\gamma)})(F)(\{(h_e,g_e)\}_{e\in E(\gamma)}) = F(\{(\alpha_{g^{-1}_{e(1)}}(h_e), g^{-1}_{e(1)}g_e g_{e(0)})\}_{e\in E(\gamma)}).$$

A non-commutative analog of Γ

Construct an inductive system of C^*-algebras $\{\mathfrak{A}_\gamma\}_\gamma$, $\mathfrak{A} = \varinjlim_\gamma \mathfrak{A}_\gamma$.

- First try: $\mathfrak{A}_\gamma = (C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|} \cong \mathcal{K}(L^2(G^{|E(\gamma)|}))$
  $\rightarrow$ Does not work (non-unital).
- Second try: $\mathfrak{A}_\gamma = M((C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|}) \cong \mathcal{B}(L^2(G^{|E(\gamma)|}))$
  $\rightarrow$ Works and has nice extension properties:
  (a) Unique extension of morphisms,
  (b) Embedding of $C(G^{|E(\gamma)|})$ and $G^{|E(\gamma)|}$,
  (c) Recovery of states on $(C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|}$ as strictly-continuous states (normal states) of $\mathfrak{A}_\gamma$.

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Constructions in $1+1$ dimensions and the infinite tensor product

Combining Jones' construction with lattice-gauge theory

General considerations

- Construct two compatible functors:

$$\Phi : \mathcal{D} \longrightarrow \mathcal{H}\text{ilb},$$
$$\Psi : \mathcal{D} \longrightarrow C^*\text{-Alg},$$

for a category $\mathcal{D}$ of binary planar forests:

$$\text{ob}(\mathcal{D}) = \{\text{binary planar trees}\},$$
$$\text{hom}(\mathcal{D}) = \{\text{binary planar forests}\} \times \{\text{permutations of leaves}\}.$$

→ Φ, Ψ are fixed by specifying them on ob$(\mathcal{D})$, the basic forest (Y, e), and the basic transposition $(||, \tau)$.

→ Binary planar trees correspond to standard dyadic partitions of $[0,1]$ (dyadic one-dimensional graphs).

→ $G_{\mathcal{D}}$ – the group of fractions of $\mathcal{D}$ – is isomorphic to Thompson's groups V. $G_{\mathcal{D}}$ acts naturally on $\mathcal{D}$.

Constructions in $1+1$ dimensions and the infinite tensor product

Combining Jones' construction with lattice-gauge theory

Lattice-gauge theory on a space-time cylinder

Because ob($\mathcal{D}$) corresponds to dyadic one-dimensional graphs, (Φ, Ψ) can be modeled on the inductive system $\{\alpha_{\gamma'\gamma} : \mathfrak{A}_\gamma \to \mathfrak{A}_{\gamma'}\}_{\gamma,\gamma'}$:

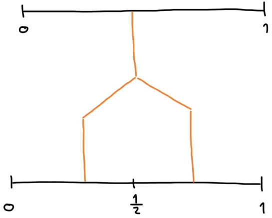
- Choose any compact group G.

$$
\begin{aligned}
\Phi(t) &= L^2(G)^{\hat{\otimes} n(t)} = \mathcal{H}_t,\ t \in \mathrm{ob}(\mathcal{D}), n(t) \text{ – number of leaves},\\
\Phi(\mathsf{Y}, e) &= R,\ (R\psi_1)(g, g') = \psi_1(gg'),\\
\Phi(||, \tau) &= U_\tau,\ (U_\tau\psi_2)(g, g') = \psi_2(g', g).
\end{aligned}
$$

$$
\begin{aligned}
\Psi(t) &= \mathcal{B}(L^2(G))^{\overline{\otimes} n(t)} = \mathfrak{A}_t,\ t \in \mathrm{ob}(\mathcal{D}),\\
\Psi(\mathsf{Y}, e) &= \tilde{R},\ \tilde{R}(a_1) = U_{\mathsf{L}}(a_1 \otimes 1)U_{\mathsf{L}}^*,\ (U_{\mathsf{L}}\psi_2)(g, g') = \psi_2(gg', g')\\
\Psi(||, \tau) &= Ad_{U_\tau}.
\end{aligned}
$$

Constructions in $1+1$ dimensions and the infinite tensor product

Combining Jones' construction with lattice-gauge theory



Construction of CFT data

A local C^*-algebra $\mathfrak{A}(I)$ is given as inductive limit over dyadic partitions of $I \subset [0,1]$:

$$\mathfrak{A}(I) = \{[\tfrac{t}{a}] : t \in \mathrm{ob}(\mathcal{D}),\ a \in \overline{\bigotimes}_{J \in P_t(I)} \mathfrak{A}_J \otimes 1\},$$

$P_t(I)$ is the partition given by t subordinate to I. $\mathfrak{A}_J$ is the algebra corresponding to the leaf in J.

- $\mathfrak{A} = \mathfrak{A}([0,1]) = \varinjlim_t \mathfrak{A}_t$, $\mathcal{H} = \varinjlim_t \mathcal{H}_t$,
- $\mathcal{A} = \mathfrak{A}''$, $\mathcal{A}(I) = \mathfrak{A}(I)''$.

Constructions in $1+1$ dimensions and the infinite tensor product

Combining Jones' construction with lattice-gauge theory

Some properties of $\{\mathcal{A}(I)\}_{I\subset[0,1]}$

- $[\mathcal{A}(I),\mathcal{A}(J)] = \{0\}$ if $I\cap J=\emptyset$,
- $g\in G_{\mathcal{D}} : \rho_g(\mathcal{A}(I)) = \mathcal{A}(gI)$,
- $g\in G_{\mathcal{D}} : \omega_\infty\circ\rho_g = \omega_\infty$.

Observations

(1) $\rho : G_{\mathcal{D}} \longrightarrow \mathrm{Aut}(\mathcal{A})$ is not strongly continuous in the induced topology of $\mathrm{Diff}(S^1)$.

(2) There is a natural equivalence

$$\eta : \Psi_{\mathsf{triv}} \longrightarrow \Psi,$$
$$\Psi_{\mathsf{triv}}(\mathsf{Y},e) = \tilde{R}_{\mathsf{triv}},\ \ \tilde{R}_{\mathsf{triv}}(a_1) = a_1\otimes 1.$$

→ $\mathcal{A}$ is isomorphic to an infinite tensor product of $\mathfrak{A}_{|}$.

→ But, the natural equivalence does not extend to an equivalence of nets.

Powers factors

The natural equivalence $\eta : \Psi_{\text{triv}} \longrightarrow \Psi$ suggests to look for tensor-product states on $\mathcal{A}$:

- For $G = \mathbb{Z}_2$, the family of states

$$\omega_{t,\lambda} = \omega_\lambda^{\otimes n(t)},\ \mathfrak{A}_t = M_2(\mathbb{C})^{\otimes n(t)},$$

$$\omega_\lambda(\ .\ ) = \mathrm{tr}(T_\lambda\ .\ ),\ T_\lambda = \frac{1}{2(1+\lambda)}\begin{pmatrix} 1+\lambda & 1-\lambda \\ 1-\lambda & 1+\lambda \end{pmatrix},\ \lambda \in [0,1],$$

  is consistent.

$\rightarrow$ $\mathcal{A}_\lambda$ is of type III$_\lambda$, $\lambda \in (0,1)$ (Powers factors, heat kernel states ($\beta = -\ln\lambda$)).

$\rightarrow$ $\mathcal{A}_0$ is of type I$_\infty$ (Ashtekar-Isham-Lewandowski state).

$\rightarrow$ $\mathcal{A}_0$ is of type II$_1$ (tracial state).

Construction of states and type classification of algebras

Observations

(1) Identify Powers' states as heat-kernel states.

$\rightarrow$ Allows for generalization to compact Lie groups.

(2) The state-consistency condition has an interpretation as renormalization group equation

$\rightarrow$ Asymptotic freedom for YM$_2$.

Hamiltonian YM$_2$ on $\mathbb{R} \times S^1_L$

Consider the Kogut-Susskind Hamiltonian on complete dyadic trees of depth N:

$$H_N = \frac{g_N}{2a_N} \sum_{n=1}^{2^{N-1}} 1 \otimes ... \otimes \Delta_G^{(n)} \otimes ... \otimes 1, \qquad a_N = \frac{L}{2^{N-1}}. \text{ (lattice spacing)}$$

$\rightarrow$ No magnetic terms in one spatial dimension.

Consider the β-KMS states associated with H_N:

$$\omega_\beta^{(N)} = (\omega_\beta^{(1)})^{\otimes n}, \qquad \omega_\beta^{(1)}(\ .\ ) = Z_\beta(a_1^{-1} g_1^2)^{-1} \,\mathrm{tr}(\exp(-\beta H_1)\ .\ ).$$

Construction of states and type classification of algebras

YM$_2$ on a space-time cylinder

State consistency

The requirement that the the β-KMS states are consistent

$$\omega_\beta^{(N)} \circ \alpha_{N-1}^N = \omega_\beta^{(N-1)},$$

leads to:

$$g_{N-1} = 2g_N^2 \Rightarrow \frac{g_N^2}{a_N} = \frac{g_1^2}{L} = \underbrace{g_0^2}_{\text{bare coupling}} L.$$

→ The state on the field algebra $\mathcal{A}_\beta$ has a Thompson-group symmetry (discrete CFT).

Observables

Implementing gauge-invariance, i.e. constructing $\mathcal{A}_\beta^{\mathcal{G}}$, $\mathcal{H}_\beta^{\mathcal{G}}$, gives

$$\mathcal{H}_\beta^{\mathcal{G}} = L^2(G)^{Ad_G},\ H = -\tfrac{1}{2}g_0^2 L\Delta_G,$$

as expected. The Hamiltonian and the "area law" can be read of from the "state sum"

$$Z_\beta(a_1^{-1}g_1^2) = \sum_{\pi\in\hat{G}} d_\pi\ e^{-\frac{\beta}{2}g_0^2 L\lambda_\pi}.$$

Thank you for your attention!