QFT in $1+1$ de Sitter spacetime and its Minkowski scaling limits

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- De Sitter space

$$dS_r \doteq \left\{ x \in \mathbb{R}^{1+2} \mid x \cdot x = x_0^2 - x_1^2 - x_2^2 = -r \right\}, \quad dS = dS_1,$$

- Wedges: set $W_1 \doteq \{x \in dS \mid x_2 > |x_0|\}$,

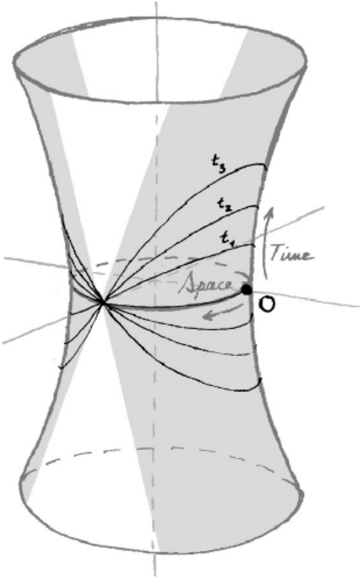
$$W = \Lambda W_1 \subset dS, \qquad \Lambda \in SO_0(1,2).$$

  The set of all wedges is denoted by $\mathcal{W}$.

- Boosts

$$\Lambda_W(t) = \Lambda \Lambda_1(t) \Lambda^{-1}, \quad \Lambda_1(t) \doteq \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

I. QFT in dS_{1+1} (2) – de Sitter wedge W_1 and opposite wedge



- $\Lambda_W(t)W = W$, $t \in \mathbb{R}$, and, for all $t \in \mathbb{R}$,

$$\Lambda_{\Lambda' W}(t) = \begin{cases} \Lambda'\Lambda_W(t)\Lambda'^{-1} & \text{if } \Lambda' \in SO_0(1,2)\ , \\ \Lambda'\Lambda_W(-t)\Lambda'^{-1} & \text{if } \Lambda' \in O_+^{\downarrow}(1,2)\,. \end{cases}$$

- Rotations

$$\alpha \mapsto R_0(\alpha) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \alpha \in [0, 2\pi)\,.$$

- Horospheric Translations

$$q \mapsto D(q) \doteq \begin{pmatrix} 1 + \frac{q^2}{2} & q & \frac{q^2}{2} \\ q & 1 & q \\ -\frac{q^2}{2} & -q & 1 - \frac{q^2}{2} \end{pmatrix}, \quad q \in \mathbb{R}\,.$$

- rotations and boosts generate $SO_0(1,2)$
- almost every element $g \in SO_0(1,2)$ can be written *uniquely* in the form

$$g = \Lambda_2(s) P^k \Lambda_1(t) D(q) \qquad \text{with } k = 0 \text{ or } k = 1\,,$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(Exceptions: $g = R_0(\pm\frac{\pi}{2})\Lambda_1(t')D(q')$) [Hannabus (1971)]

- for $|q|$ small:

$$D(q)\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} - R_0(-q)\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} = O(q^2)$$

An algebraic QFT on dS_r is given by:

[Bros, Epstein, Moschella (1998); Borchers, Buchholz (1999)]

- a Hilbert space $\mathcal{H}_r$ carrying a continuous unitary representation $U_r(g)$, $g \in SO_0(1,2)$
- a unit vector $\Omega_r \in \mathcal{A}_r$ which is invariant:

$$U_r(g)\Omega_r = \Omega_r \quad (g \in SO_0(1,2))$$

- a family of von Neumann algebras $\mathcal{A}_r(W)$, $W \in \mathcal{W} = \mathcal{W}_r$, acted on covariantly by the group:

$$\alpha_g^{(r)}(\mathcal{A}_r(W)) = \mathcal{A}_r(g(W)) \,, \qquad \alpha_g^{(r)}(A) = U_r(g)AU_r(g)^*$$

- Ω_r is a standard unit vector for any $\mathcal{A}_r(W)$ and $(\Omega_r, \,.\,\Omega_r)_r$ restricts on $\mathcal{A}_r(W)$ to a KMS state for the boosts $t \mapsto \Lambda_W(t/r)$ at inverse temperature $\beta_r = 2\pi r$.
- $J_{W_r}\mathcal{A}_r(W_r)J_{W_r} = \mathcal{A}_r(W_r')$ where W_r' is the opposite wedge = causal complement of W_r. (J_{W_r} = modular conjugation)
- Defining for any double cone $O \subset dS_r$,

$$\mathcal{A}_r(O) = \bigcap_{W \supset O} \mathcal{A}_r(W)\,,$$

  the family of the $\mathcal{A}_r(O)$ fulfills isotony, locality and covariance w.r.t. the action of $SO_0(1,2)$.

Quantum field models on dS_r fulfilling the assumptions of $(\mathcal{H}_r, \mathcal{A}_r, U_r, \Omega_r)$:

- the CCR quantized scalar field with field eqn $(\Box + \xi R + m_0^2)\phi = 0$ with dS_r "vacuum state" ω_r [Bros, Moschella (1996)]
- interacting fields: $\mathcal{P}(\phi)_2$ [Barata, Jäkel, Mund, (*to appear in Memoirs of AMS*); Jäkel, Mund (2018)]; earlier work e.g. [Figari, Hoegh-Kron, Nappi (1975)]

Construction of interacting QFT on dS_{1+1} has several attractive features:

$(+)$ QFT can be constructed (from "Cauchy data fields") in the "vacuum" GNS Hilbert space representation of the free field; interacting QFT field algebras act in that Hilbert space

$(+)$ unitary representation of rotations on "Cauchy data fields" is the same for interacting fields as for free fields

$(+)$ therefore, at an abstract level, the construction of an interacting QFT such as $\mathcal{P}(\phi)_2$ on dS_{1+1} amounts to identifying all unit vectors $\tilde{\Omega}$ in the "vacuum" GNS Hilbert space of the free field so that

- $(\star)$ $\tilde{\Omega}$ is invariant under rotations
- $(\star)$ $\tilde{\Omega}$ is a standard vector for the Cauchy data wedge algebras
- $(\star)$ the associated modular objects act geometrically correctly as in the conditions on $(\mathcal{H}_r, \mathcal{A}_r, U_r, \Omega_r)$

- For each dS_r, $r \geq 1$, a QFT is given:

$$(\mathcal{H}_r, \mathcal{A}_r, U_r, \Omega_r)$$

  where $\mathcal{A}_r \subset \mathrm{B}(\mathcal{H}_r)$ is the von Neumann algebra generated by all the $\mathcal{A}_r(W)$, $W \in \mathcal{W}_r$.
- We assume the QFT at each r to be “the same” in a suitable sense
- We define a **scaling algebra** in the spirit of [Buchholz, RV (1995)] which provides a framework for investigating the limiting behaviour of the QFTs as $r \to \infty$.

  Expect: In the limit $r \to \infty$, the theories should approximate a QFT on 1 + 1 dimensional Minkowski spacetime.

- $\boldsymbol{A}$ is the unital C^* of all families $\underline{A} = (\underline{A}_r)_{1 \leq r < \infty}$ where

$$\underline{A}_r \in \mathcal{A}_r \quad \text{and} \quad ||\underline{A}|| = \sup_r ||\underline{A}_r||_r < \infty$$

  The algebraic operations are pointwise defined, i.e. for any r.
- There is an action of $SO_0(1,2)$ on $\boldsymbol{A}$:

$$\underline{\alpha}_g(\underline{A})_r = \alpha_g^{(r)}(\underline{A}_r)$$

- Let $\boldsymbol{G}$ be the group of all (continuous) functions

$$\boldsymbol{g} : [1, \infty) \to SO_0(1,2)$$

  Then also $\boldsymbol{G}$ acts on $\boldsymbol{A}$ by automorphisms:

$$\underline{\alpha}_{\boldsymbol{g}}(\underline{A})_r = \alpha_{\boldsymbol{g}(r)}^{(r)}(\underline{A}_r)$$

Let N be a neighbourhood of the unit element 1 in $SO_0(1,2)$.

$$\textbf{Notation}: \quad \boldsymbol{g} \in N \quad \text{if} \quad \boldsymbol{g}(r) \in N \;\; \text{for all} \;\; r$$

- Define $\underline{\mathcal{A}}$ as the C^* subalgebra of $\boldsymbol{A}$ formed by all $\underline{A}$ such that

$$\sup_{\boldsymbol{g}\in N} ||\underline{\alpha}_{\boldsymbol{g}}(\underline{A}) - \underline{A}|| \to 0 \quad (N \to \{1\})$$

This is a large subalgebra of $\boldsymbol{A}$: For any $\underline{A} \in \boldsymbol{A}$ and $f \in L^1(SO_0(1,2), d\mathrm{H})$ ($d\mathrm{H}$ = Haar measure), $\underline{A}_f$ defined by

$$(\underline{A}_f)_r = \int f(g)\, \alpha_g^{(r)}(\underline{A}_r)\, d\mathrm{H}(g)$$

has the required continuity property.

Note: If $\underline{A} \in \underline{\mathcal{A}}$ and $f \in L^1(SO_0(1,2), d\mathrm{H})$ then $\underline{A}_f \in \underline{\mathcal{A}}$.

Family of **lifted states** $(\underline{\omega}^{(r)})_{r\geq 1}$ on $\underline{\mathcal{A}}$:

$$\underline{\omega}^{(r)}(\underline{A}) = \omega_r(\underline{A}_r)\,, \quad \omega_r(A) = (\Omega_r, A\Omega_r)_r$$

It holds that

$$\underline{\omega}^{(r)} \circ \underline{\alpha}_{\boldsymbol{g}} = \underline{\omega}^{(r)}$$

The family $(\underline{\omega}^{(r)})_{r\geq 1}$ possesses **weak-$*$ limit points** as $r \to \infty$:
There are **scaling limit states** $\underline{\omega}^{(\infty)}$ on the C^* algebra $\underline{\mathcal{A}}$ arising as

$$\underline{\omega}^{(\infty)}(\underline{A}) = \lim_{\kappa} \underline{\omega}^{(r_\kappa)}(\underline{A})$$

for some **generalized sequence** $(\lambda_\kappa)_{\kappa\in K}$ (K is an ordered set)
with $\lim_\kappa r_\kappa = \infty$.

- There may occur **different** scaling limit states depending on the generalized sequence $(r_\kappa)_{\kappa \in K}$; e.g. the GNS representations of different scaling limit states might be disjoint (not unitarily equivalent or quasiequivalent).
- Obviously (from the properties of the lifted states) for any scaling limit state :

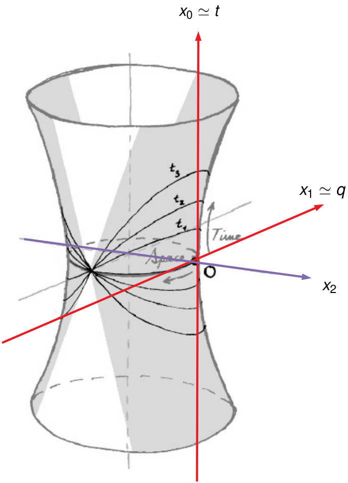
$$\underline{\omega}^{(\infty)} \circ \underline{\alpha}_{\boldsymbol{g}} = \underline{\omega}^{(\infty)}$$

- In the GNS representation $(\mathcal{H}^{(\infty)}, \pi^{(\infty)}, \Omega^{(\infty)})$ of any scaling limit state $\underline{\omega}^{(\infty)}$:

  $U_{\boldsymbol{g}}^{(\infty)}(\pi^{(\infty)}(\underline{A}))\Omega^{(\infty)} = \pi^{(\infty)}(\underline{\alpha}_{\boldsymbol{g}}(\underline{A}))\Omega^{(\infty)}$ is a unitary group representation which is continuous: For any $\psi \in \mathcal{H}^{(\infty)}$,

$$\sup_{\boldsymbol{g} \in N} ||U_{\boldsymbol{g}}^{(\infty)}\psi - \psi||_{\mathcal{H}^{(\infty)}} \to 0 \quad (N \to \{1\})$$

All dS_r are shifted by $-r$ along the x_2 axis so that the $x_2 = 0$ hyperplane is the common tangent plane of all the $dS^{(r)} = dS_r - r\vec{e}_2 = \mathtt{T}_r(dS_r)$



- $\mathtt{T}_r \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 - r \end{pmatrix}$
- 2-dim Minkowski spacetime is identified with the $x_2 = 0$ hyperplane in $\mathbb{R}_{1+2}$,
  in the limit as $r \to \infty$, the right wedge $W_1^{(r)}$ of $dS^{(r)}$ approximates the $x_2 = 0$ hyperplane
- identify $SO_0(1,2) \xrightarrow{\mathrm{Ad}\,\mathtt{T}_r} \mathrm{Iso}_0(dS^{(r)})$
- set up the scaling algebra $\underline{\mathcal{A}}$ as before, however

$$(\star) \quad (\mathcal{H}_r, \mathcal{A}_r, U_r, \Omega_r) \text{ is QFT on } dS^{(r)}$$

$$(\star) \quad U_r \text{ is unitary representation of } \mathrm{Iso}_0(dS^{(r)})$$

Conformal embedding of $\mathbb{R}_{1+1}$ into $W_1^{(r)}$

$$\chi_r \begin{pmatrix} t \\ q \\ 0 \end{pmatrix} = \begin{pmatrix} r\frac{\sinh(t/r)}{\cosh(q/r)} \\ r\tanh(q/r) \\ r\frac{\cosh(t/r)}{\cosh(q/r)} - r \end{pmatrix}$$

Embedding of $\mathcal{P}_+^\uparrow(2)$ into $\mathrm{Iso}_0(dS^{(r)})$

$$\boldsymbol{g}_L(r) = \mathtt{T}_r\Lambda_2(s)\Lambda_1(t/r)D(q/r)\mathtt{T}_r^{-1}$$

for

$$L \begin{pmatrix} t' \\ q' \\ 0 \end{pmatrix} = \Lambda_2(s) \begin{pmatrix} t' + t \\ q' + q \\ 0 \end{pmatrix}$$

"$\mathbb{R}_{1+1}$ conformally local" scaling algebras:

$\underline{\mathcal{A}}(O)$ is defined as the C^* subalgebra of all $\underline{A} \in \underline{\mathcal{A}}$ with

$$\underline{A}_r \in \mathcal{A}_r(\chi_r(O))$$

for any double cone $O \subset \mathbb{R}_{1+1}$.

Observation

It holds that $\underline{\mathcal{A}}(O_1)$ and $\underline{\mathcal{A}}(O_2)$ commute if O_1 and O_2 are causally separated since every QFT on $dS^{(r)}$ fulfills the locality condition and the embedding χ_r is conformal, therefore preserves causality relations.

Proposition 1

(i) $$\chi_r \begin{pmatrix} t \\ q \\ 0 \end{pmatrix} \xrightarrow[r\to\infty]{} \begin{pmatrix} t \\ q \\ 0 \end{pmatrix}$$

(ii) $$\boldsymbol{g}_L(r)\boldsymbol{g}_{L'}(r)\boldsymbol{g}_{(LL')^{-1}}(r) \begin{pmatrix} t \\ q \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ q \\ x_2 \end{pmatrix}$$

In the limit $r \to \infty$, $\boldsymbol{g}_L(r)$ furnishes a group contraction from $SO_0(1,2)$ to $\mathcal{P}_+^\uparrow(2)$ in the sense of [Mickelsson and Niederle (1972)]

(iii) $$\boldsymbol{g}_L(r)\chi_r \begin{pmatrix} t \\ q \\ 0 \end{pmatrix} \xrightarrow[r\to\infty]{} L \begin{pmatrix} t \\ q \\ 0 \end{pmatrix}$$

Proposition 2

Let $\underline{\omega}^{(\infty)}$ be a scaling limit state of $\underline{\mathcal{A}}$, with GNS representation $(\mathcal{H}^{(\infty)}, \pi^{(\infty)}, \Omega^{(\infty)})$

(I) The family of C^* algebras $\mathcal{M}(O) = \pi^{(\infty)}(\underline{\mathcal{A}}(O))$ indexed by the double cones $O \subset \mathbb{R}_{1+1}$ fulfills isotony and locality (spacelike commutativity)

(II)

$$\pi^{(\infty)} \circ \underline{\alpha}_{\boldsymbol{g}_L} \circ \underline{\alpha}_{\boldsymbol{g}_{L'}} = \pi^{(\infty)} \circ \underline{\alpha}_{\boldsymbol{g}_{LL'}}$$

$$\pi^{(\infty)}(\underline{\alpha}_{\boldsymbol{g}_L}(\underline{\mathcal{A}}(O)) = \pi^{(\infty)}(\underline{\mathcal{A}}(LO))$$

Hence, by invariance of $\underline{\omega}^{(\infty)}$ under the $\underline{\alpha}_{\boldsymbol{g}}$, there is a unitary group representation $U(L)$, $L \in \mathcal{P}_+^\uparrow(2)$ on $\mathcal{H}^{(\infty)}$ so that

$$U(L)\mathcal{M}(O)U(L)^* = \mathcal{M}(LO) \quad \text{and} \quad U(L)\Omega^{(\infty)} = \Omega^{(\infty)}$$

(III) The unitary group representation $U(L)$ of $\mathcal{P}_+^{\uparrow}(2)$ fulfills the relativistic spectrum condition.

In summary: Any scaling limit theory

$$(\mathcal{H}^{\infty}, \mathcal{M}, U, \Omega^{\infty})$$

is an AQFT on $\mathbb{R}_{1+1}$ in vacuum representation.

To be addressed:

- $(*)$ Is that vacuum representation irreducible?
- $(*)$ Is the scaling limit theory non-trivial? (OK e.g. for KG-field on every $dS^{(r)}$) How does it relate to the theory at finite scale?
- $(*)$ How do we know we have the same QFT on any $dS^{(r)}$ — relation to "same physics in all spacetimes"

[Fewster, RV (2012)], cf. [Kay (1978)]