Global anomalies on Lorentzian space-times

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Global anomalies in the path integral

- Chiral $SU(2)$ doublet: Not anomalous w.r.t. infinitesimal (local) gauge trafos.
- But: Anomalous w.r.t. large (global) gauge trafos [Witten 82].
- As $\pi_4(SU(2)) = \mathbb{Z}_2$, there are compactly supported gauge trafos g that can not be deformed to the identity.
- However, one may deform A to A^g via a path A_λ of connections that are not gauge equivalent to A. Along such a path, the fermion path integral

$$\left[\int \mathrm{d}\psi \mathrm{d}\bar{\psi} \exp(\bar{\psi} i \not{D}_{A_\lambda} \psi)\right]^{\frac{1}{2}} = \left[\det i\not{D}_{A_\lambda}\right]^{\frac{1}{2}}$$

  changes sign as A is varied to A^g (mod 2 index theorem).
- This implies that the full partition function

$$Z = \int \mathrm{d}A \left[\det i\not{D}_A\right]^{\frac{1}{2}} \exp\left(-\frac{1}{2g_{YM}^2}\int \mathrm{tr}\, F \wedge \star F\right)$$

  vanishes, as the contributions from A and A^g always cancel.
- The theory is thus inconsistent.
- Non-perturbative effect, not visible in perturbation theory around single background.

Riemannian vs. Lorentzian

- The computations of global anomalies involve fermions in background fields in Riemannian signature.
- No clear relation to Lorentzian signature.
- What is an appropriate condition for global anomalies in Lorentzian signature (based on free fermions in non-trivial backgrounds)?
- How does a global anomaly render a theory inconsistent?

The framework (I)

- As in the path integral framework, we formulate a criterion for global anomalies based on free chiral fermions in generic gauge backgrounds.
- Gauge backgrounds described by principal bundle connection $\bar{\mathcal{A}}$.
- Two backgrounds $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$ differ by Lie-algebra valued one-form $A = \bar{\mathcal{A}} - \bar{\mathcal{A}}'$.
- Locally covariant field theory [Hollands, Wald 01; Brunetti, Fredenhagen, Verch 03] adapted to the gauge theory setting [Z. 14]: Local covariance also w.r.t. principal bundle morphisms.
- Fields provide a consistent assignment of observables to different backgrounds. Example: The current

$$j_{\bar{\mathcal{A}}}(A) = \left\langle \tfrac{\delta}{\delta \bar{\mathcal{A}}} S, A \right\rangle = - \int \bar{\psi} \not{A} \psi \, vol$$

  defined by point-splitting w.r.t. the Hadamard parametrix.
- No local anomalies, i.e., the current is conserved

$$\bar{\delta} j_{\bar{\mathcal{A}}}(\Lambda) \doteq j_{\bar{\mathcal{A}}}(\bar{\mathrm{d}} \Lambda) = 0. \qquad \text{(CC)}$$

  It is then unique up to charge renormalization [Z. 14]

$$j_{\bar{\mathcal{A}}} \to j_{\bar{\mathcal{A}}} + \lambda \bar{\delta} \bar{F}.$$

The framework (II)

- When two backgrounds $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$ differ only in a compact region, there is a natural isomorphism of the corresponding algebras, the retarded variation

$$\tau^{\mathrm{r}}_{\bar{\mathcal{A}},\bar{\mathcal{A}}'} : \mathfrak{A}(\bar{\mathcal{A}}') \to \mathfrak{A}(\bar{\mathcal{A}}).$$

- It acts trivially on observables localized in the past of $\mathrm{supp}(\bar{\mathcal{A}}' - \bar{\mathcal{A}})$.
- Perturbative agreement (PA) [Hollands, Wald 05] is the requirement that it should not matter whether one puts quadratic terms in the free or interaction part of the action:

$$\tau^{\mathrm{r}}_{\bar{\mathcal{A}},\bar{\mathcal{A}}'}(\mathcal{T}_{\bar{\mathcal{A}}'}(e^F)) = \mathcal{R}_{\bar{\mathcal{A}}}(e^F; e^{ij(\bar{\mathcal{A}}'-\bar{\mathcal{A}})}) \doteq \mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(\bar{\mathcal{A}}'-\bar{\mathcal{A}})})^{-1}\mathcal{T}_{\bar{\mathcal{A}}}(e^F \otimes e^{ij(\bar{\mathcal{A}}'-\bar{\mathcal{A}})})$$

- The infinitesimal retarded variation around $\bar{\mathcal{A}}$ in the direction of A is denoted by $\delta^{\mathrm{r}}_{\bar{\mathcal{A}}}(A)$.
- (PA) can be fulfilled provided that

$$E_{\bar{\mathcal{A}}}(A_1, A_2) \doteq \delta^{\mathrm{r}}_{\bar{\mathcal{A}}}(A_1) j(A_2) - \delta^{\mathrm{r}}_{\bar{\mathcal{A}}}(A_2) j(A_1) - i[j(A_2), j(A_1)] = 0.$$

In dimension $d \leqslant 4$, (CC) implies $E_{\bar{\mathcal{A}}}(A_1, A_2) = 0$ [Z. 15].

The phase of the S matrix

- Our criterion for the occurrence of a global anomaly will be a non-trivial phase of the S matrix for $\bar{\mathcal{A}} \to \bar{\mathcal{A}}^g$. Need to fix the phase of the S matrix.
- Formally, the S matrix for $\bar{\mathcal{A}} \to \bar{\mathcal{A}}' = \bar{\mathcal{A}} + A$ is given by and fulfills

$$\begin{aligned} S_{\bar{\mathcal{A}}}(A) &= \mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(A)}) \\ &= \mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(A')})\mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(A')})^{-1}\mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(A-A')} \otimes e^{ij(A')}) \\ &= \mathcal{T}_{\bar{\mathcal{A}}}(e^{ij(A')})\mathcal{R}_{\bar{\mathcal{A}}}(e^{ij(A-A')}; e^{ij(A')}) \\ &= S_{\bar{\mathcal{A}}}(A')\tau^{\mathrm{r}}_{\bar{\mathcal{A}},\bar{\mathcal{A}}+A'}(S_{\bar{\mathcal{A}}+A'}(A-A')) \end{aligned}$$

- With the further constraints

$$S_{\bar{\mathcal{A}}}(0) = \mathbb{1}, \qquad \partial_\lambda S_{\bar{\mathcal{A}}}(A_\lambda)|_{\lambda=0} = ij_{\bar{\mathcal{A}}}(\dot{A}_0),$$

we may integrate S matrix for any path $[0,1] \ni \lambda \mapsto A_\lambda$ from 0 to A:

$$S_{\bar{\mathcal{A}}}(A) = \bar{P}\exp\left(i\int_0^1 \tau^{\mathrm{r}}_{\bar{\mathcal{A}},\bar{\mathcal{A}}+A_\lambda}(j_{\bar{\mathcal{A}}+A_\lambda}(\dot{A}_\lambda))\mathrm{d}\lambda\right) \qquad \text{(PO)}$$

- Path independence is equivalent to $E = 0$.
- Unique up to

$$S_{\bar{\mathcal{A}}}(A) \to \exp\left(i\lambda\int\left[L_{YM}(\bar{\mathcal{A}}+A) - L_{YM}(\bar{\mathcal{A}})\right]\right)S_{\bar{\mathcal{A}}}(A).$$

Hilbert space representation

- A representation $\bar{\pi} : \mathfrak{A}(\bar{\mathcal{A}}) \to \mathrm{End}(\bar{\mathcal{H}})$ naturally induces representations

$$\pi_A \doteq \bar{\pi} \circ \tau^{\mathrm{r}}_{\bar{\mathcal{A}},\bar{\mathcal{A}}+A} : \mathfrak{A}(\bar{\mathcal{A}} + A) \to \mathrm{End}(\bar{\mathcal{H}}).$$

- In the representation, (PO) reads

$$U(A, A') \doteq \pi_A(S_{\bar{\mathcal{A}}+A}(A' - A)) = \bar{P} \exp\left(i \int_0^1 \pi_{A_\lambda}(j_{\bar{\mathcal{A}}+A_\lambda}(\dot{A}_\lambda))\mathrm{d}\lambda \right),$$

  with A_λ a path from A to A'.
- Q: Is $\pi(j)$ self-adjoint? Is U well-defined and unitary?
- Assuming it is,

$$U(A, A')U(A', A'') = U(A, A''), \qquad U(A, A')^{-1} = U(A', A).$$

- Furthermore, $V(g) \doteq U((\bar{\mathcal{A}} + A)^g - \bar{\mathcal{A}}, A) = e^{i\phi_g}\mathrm{id}$ is independent of A, and thus provides a representation of the gauge group $\Gamma^\infty_c(M, P \times_{\mathrm{Ad}} G)$.
- If g is deformable to the identity, then, by (PO) and (CC), $V(g) = \mathrm{id}$.
- If $V(g) \neq \mathrm{id}$ for some g, then no gauge invariant vector, a global anomaly.
- Same topological obstructions as in the path integral formalism and similar computation via gauge non-equivalent connections.

Global anomalies in a Hamiltonian framework

- Following [Witten 82], assume that the Hilbert space is given by sections over the space of $3d$ gauge fields in temporal gauge. The gauge group is then $\mathcal{G} = C_c^\infty(\mathbb{R}^3, G)$ with homotopy group

$$\pi_1(\mathcal{G}) = \pi_4(G).$$

- Physical states are annihilated by the generators $Q(\Lambda)$ of $\mathcal{G}$.
- The non-trivial element of $\pi_1(\mathcal{G})$ must be represented by the identity, otherwise there are no physical states.
- The matter contribution to the generators is $Q_{matter}(\Lambda) = j_{\bar{\mathcal{A}}}(B)$ with

$$B^a_\mu(x) = \delta^0_\mu \Lambda^a(\vec{x})\delta(x^0).$$

- $E = 0$ ensures

$$[Q(\Lambda), Q(\Lambda')] = iQ([\Lambda, \Lambda']).$$

- In the case of a global anomaly, there are no physical states, as integrating up $Q(\Lambda)$ along a non-trivial cycle does not yield the identity.

Perturbative agreement and the Wess-Zumino consistency condition

- Assume there is a local anomaly, i.e., (CC) does not hold. Can we still obtain $E_{\bar{\mathcal{A}}}(A, A') = 0$ by giving up the requirement that j is a field?
- We fix a flat reference connection $\bar{\mathcal{A}}_0$ and specify any other background $\bar{\mathcal{A}} = \bar{\mathcal{A}}_0 + \bar{A}$ by a vector potential $\bar{A}$. Allow $j_{\bar{A}}$ to depend on $\bar{A}$. We have

$$E_{\bar{A}}(\mathrm{d}_{\bar{A}}\Lambda, \mathrm{d}_{\bar{A}}\Lambda') = \left\langle \tfrac{\delta}{\delta \bar{A}} \bar{\delta} j_{\bar{A}}(\Lambda'), \mathrm{d}_{\bar{A}}\Lambda \right\rangle - \left\langle \tfrac{\delta}{\delta \bar{A}} \bar{\delta} j_{\bar{A}}(\Lambda), \mathrm{d}_{\bar{A}}\Lambda' \right\rangle - \bar{\delta} j_{\bar{A}}([\Lambda, \Lambda']) \stackrel{!}{=} 0. \tag{WZ}$$

  This is the Wess-Zumino consistency condition.
- For $d = 4$ and flat space-time [Z. 14],

$$\bar{\delta} j_{\bar{\mathcal{A}}}(\Lambda) = \frac{i}{8\pi^2} \int \operatorname{tr} \Lambda \bar{F} \wedge \bar{F}.$$

- With [Bardeen & Zumino 84]

$$j_{\bar{A}}(A) \mapsto j_{\bar{A}}(A) + \frac{i}{24\pi^2} \int \operatorname{tr} \left[A \wedge (\bar{A} \wedge \bar{F} + \bar{F} \wedge \bar{A} - \tfrac{1}{2} \bar{A} \wedge \bar{A} \wedge \bar{A}) \right]$$

  one obtains $E_{\bar{A}}(A_1, A_2) = 0$ and the consistent anomaly

$$\bar{\delta} j_{\bar{A}}(\Lambda) = \frac{i}{24\pi^2} \int \operatorname{tr} \left[\Lambda (\mathrm{d}\bar{A} \wedge \mathrm{d}\bar{A} + \tfrac{1}{2} \mathrm{d}(\bar{A} \wedge \bar{A} \wedge \bar{A})) \right].$$

- For $G = U(1)$ and flat space-time, one can obtain (CC) and (WZ), but then $E_{\bar{A}}(A_1, A_2) \neq 0$. Hence, (PA) is stronger than (WZ).

Computation of the $SU(2)$ anomaly

- Following [Witten 83; Elitzur & Nair 84], compute $SU(2)$ anomaly by embedding $G = SU(2) \subset SU(3) = H$ with $\pi_4(H) = 0$. May connect the nontrivial $g \in \pi_4(G)$ by a path in $C_c^\infty(\mathbb{R}^4, H)$ to the identity. With (PO), the global anomaly of G is computed by integrating the consistent anomaly of H:

$$S_{\bar{A}}(\bar{A}^g - \bar{A}) = \exp\left(\frac{1}{48\pi^2}\int_0^1 \mathrm{d}\lambda \int \mathrm{tr}\left(h^{-1}\dot{h} \wedge A \wedge A \wedge A \wedge A\right)\right)$$
$$= \exp\left(\frac{1}{240\pi^2}\int_{[0,1]\times\mathbb{R}^4} h^*(\mu_H^5)\right)$$

where $h(0) = \mathrm{id}$, $h(1) = g$, $A = h^{-1}\mathrm{d}h$, and $\bar{A}$ is flat.

- h defines an element of $\pi_5(H/G)$ and $[h] \mapsto \frac{1}{240\pi^2}\int_{S^5} h^*(\mu_H^5)$ is a group homomorphism, which for the generator h_1 of $\pi_5(H)$ is normalized to

$$\frac{1}{240\pi^2}\int_{S^5} h_1^*(\mu_H^5) = 2\pi i.$$

- We have the exact sequence

$$\pi_5(H) = \mathbb{Z} \;\to\; \pi_5(H/G) = \mathbb{Z} \;\to\; \pi_4(G) = \mathbb{Z}_2 \;\to\; \pi_4(H) = 0.$$

Hence $\frac{1}{240\pi^2}\int_{S^5} h^*(\mu_H^5)$ is odd multiple of $i\pi$, so that $S_{\bar{A}}(\bar{A}^g - \bar{A}) = -\mathrm{id}$.

Summary & Outlook

Summary:

- Interpreted global anomalies in a Lorentzian setting.
- Phase of the S matrix.
- Pivotal role of perturbative agreement ($E = 0$).
- Relation of perturbative agreement and WZ consistency.

Open issues:

- Unitarity of implementers in representation.
- Effect of non-trivial topologies.